Chapter 10: Parametric And Polar Curves; Conic Sections

Summary: This chapter begins by introducing the idea of representing curves using parameters. These parametric equations of the curves can then be used to graph, find tangent lines, and find arc lengths of the given curves. Polar coordinates and their relationship to Cartesian or rectangular coordinates are then described. Polar coordinates may also be used as a particular parameterization of curves. Then tangent lines, arc length, and area may be found using polar coordinates as the parameterization of the curve. The latter half of the chapter discusses conic sections and how they may be graphed. Then the idea of conic sections is extended by representing these curves using polar coordinates.

OBJECTIVES: After reading and working through this chapter you should be able to do the following:

1. Represent a function as a parametric curve (§10.1).
2. Find tangent lines of curves that are either defined parametrically (§10.1) or defined using polar coordinates (§10.3).
3. Find the arc length of a parametric curve (§10.1) or a polar curve (§10.3).
4. Convert between rectangular and polar coordinates (§10.2).
5. Graph functions in polar coordinates (§10.2).
6. Find the area between curves that have been defined in polar coordinates (§10.3).
7. Identify different conic sections and their graphs (§10.4).
8. Rotate conic sections and determine their resulting quadratic equations (§10.5).
9. Describe conic sections and their graphs using polar coordinates (§10.6).
10.1 Parametric Equations; Tangent Lines And Arc Length For Parametric Curves

PURPOSE: To discuss tangent lines and arc length of parametric curves and curves in polar coordinates.

One of the main ideas in this section is to come up with functions for both the $x$ and $y$ coordinates of a curve. In this way, curves that are generally not considered as functions may be graphed quite easily. For example, this gives a great way to graph the inverse of a function. If $x = f(t)$ and $y = g(t)$ then the inverse can be graphed by reversing the definitions. $x = g(t)$ and $y = f(t)$ would result in the graph of the inverse. The parameter $t$ used in the definitions of $x$ and $y$ also results in the parametric curve having a definite orientation. If $t$ is followed in a positive direction, for example, then this will take you in one direction along a curve while a negative direction for $t$ would take you in the opposite direction.

Tangent lines were first introduced in Chapters 1 and 2 where the idea of a derivative was first developed. The same kinds of tangent lines to curves are discussed in this section. The new approach taken here is to represent the curves parametrically by defining a function to each coordinate of a point on the curve, or as polar curves.

To find a tangent line to a curve requires that the slope of the curve at a point be found. This is done using the derivative of the curve. In other words, the slope of a tangent line is equivalent to $dy/dx$. If $x = f(t)$ and $y = g(t)$ then this slope can be found by dividing $g'(t)$ by $f'(t)$. One way to remember this is to think of these slopes as differentials and divide out some of the terms. For example,

$$\frac{g'(t)}{f'(t)} = \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{dy}{dx}$$

which is the slope of the tangent line.

Some special types of tangent lines are horizontal tangents and vertical tangents. **Horizontal tangents** occur when $dy/dt = 0$ and $dx/dt \neq 0$. **Vertical tangents** occur at a point on the curve where $dy/dt \neq 0$ and $dx/dt = 0$. If both $dy/dt = 0$ and $dx/dt = 0$ then this is a **singular point** on the curve that does not have a tangent line.

**CAUTION:** To have a vertical tangent at a point does require that the curve be defined at that point. For example, vertical asymptotes do not have vertical tangents.

Calculating **arc length** for a parametric curve is done using the arc length formula (see §6.4) but modified to include both $x'(t)$ and $y'(t)$ in the equation.

$$L = \int_{t=a}^{t=b} \sqrt{(x'(t))^2 + (y'(t))^2} \, dt$$
Checklist of Key Ideas:

☐ tangent lines
☐ infinite slope, vertical tangents
☐ singular points
☐ arc length of a parametric curve
☐ parametric first and second derivatives

10.2 Polar Coordinates

**PURPOSE:** To introduce polar coordinates and to graph curves that are represented using polar coordinates.

Until this point in the book, coordinates have been measured in a rectangular fashion. So any given point is identified by a horizontal value, \( x \), and a vertical value, \( y \). Polar coordinates offer a different way to identify each point. This new identification can be seen by drawing a line from the origin (called the pole) out to the point. The point is then identified by how far it is from the origin (this is the \( r \)-coordinate) and the angle that the line makes with the positive \( x \)-axis (called the polar axis). This angle is the \( \theta \)-coordinate. Angles are measured using radian measure.

**CAUTION:** The angle \( \theta \) should be given in radian measure where 1 radian = 180/\( \pi \) degrees.

Using trigonometric formulas it can be seen that if the same point has polar coordinates of \((r, \theta)\) then the corresponding rectangular coordinates, \((x,y)\), will be determined by the equations

\[
x = r \cos \theta \quad \text{and} \quad y = r \sin \theta.
\]

Following similar reasoning, if a point is designated as \((x,y)\) in rectangular coordinates then the point in polar coordinates \((r, \theta)\) can be found by the following equations:

\[
r^2 = x^2 + y^2 \quad \text{and} \quad \tan \theta = y/x.
\]

**CAUTION:** Polar coordinates of a point are not unique.

Polar coordinates for a point are not unique. The reason for this is that there are many values of \( \theta \) that can be chosen so that \( \tan (\theta) = y/x \) and so there are many ways to choose \( \theta \). For example, the rectangular point \((2,2)\) can be identified by any of the following polar points: \((2\sqrt{2}, \pi/4)\), \((-2\sqrt{2}, -3\pi/4)\), \((2\sqrt{2}, 9\pi/4)\), or \((-2\sqrt{2}, 5\pi/4)\).
IDEA: Graphing polar curves involves always thinking of points by drawing a line from the origin to the point.

Graphing in polar coordinates is not difficult although it is definitely different than graphing in rectangular coordinates. Graphing involves the same process as identifying points. Any point on a graph should be thought of using a straight line to the origin. Then the length of the line and the angle of the line are used to determine  

Many polar graphs are symmetric about the x-axis, the y-axis or the origin. Symmetry in polar graphs often occurs because of the repetitive nature of angles. For example, the graph of  

IDEA: Symmetry of polar curves can be tested for in three ways:

| 1. | symmetric about the x-axis if replacing $\theta$ with $-\theta$ gives an equivalent equation |
| 2. | symmetric about the y-axis if replacing $\theta$ with $\theta + \pi$ gives an equivalent equation |
| 3. | symmetric about the origin if replacing $\theta$ with $\theta - \pi$ (or replacing $r$ with $-r$) gives an equivalent equation |

IDEA: When checking for symmetry, it can be helpful to remember two important properties of sine and cosine.

1. **sine is odd** $\rightarrow \sin(-\theta) = -\sin(\theta)$
2. **cosine is even** $\rightarrow \cos(\theta) = \cos(-\theta)$

The two trigonometric functions, sine and cosine, are odd and even functions respectively. These can be useful ideas when testing to see whether a polar graph will be symmetric or not as many of them contain either sine or cosine in some fashion.

Many graphs of functions are introduced in this section. If recognizing all of the different graphs is important then it may be helpful to write each different example on note cards that can be quickly reviewed. Here are some of the main graphs and their names.

1. lines through the pole: $\theta = \theta_0$
2. circles: $r = a$, $r = 2a\cos\theta$, $r = 2a\sin\theta$
3. rose curves: $r = a\sin(n\theta)$ or $r = a\cos(n\theta)$ where $n$ is a positive integer
4. lemniscates: $r^2 = a^2\cos(2\theta)$ or $r^2 = a^2\sin(2\theta)$
5. limaçons: \( r = a \pm b \sin \theta \) or \( r = a \pm b \cos \theta \) where \( a \) and \( b \) are positive. Here are some types of limaçons:
   \( \rightarrow \) with an inner loop: \( a/b < 1 \)
   \( \rightarrow \) a cardioid (special type of limaçon: \( a = b \))
   \( \rightarrow \) dimpled: \( 1 < a/b < 2 \)
   \( \rightarrow \) convex: \( a/b \geq 2 \)

6. spirals: there are several different types of these but they all have the common feature of circling around the pole in larger and larger paths. Here are some common types with \( a \) and \( b \) positive.
   \( \rightarrow \) Archimedean: \( r = a \theta \)
   \( \rightarrow \) parabolic: \( r = a \sqrt{\theta} \)
   \( \rightarrow \) logarithmic: \( r = ae^{b \theta} \)
   \( \rightarrow \) Lituus: \( r = a/\sqrt{\theta} \)
   \( \rightarrow \) hyperbolic: \( r = a/\theta \)

There are many varieties of polar curves that can be graphed. To get a handle on what a graph may look like, some common points to try and find are where the curve may intersect with the \( x \)- and \( y \)-axes. For example, the angle \( \theta = 0 \) corresponds to the positive \( x \)-axis. When this is substituted into an expression, a value for \( r \) may be found to determine where the curve touches the \( x \)-axis. Other angles that may be checked easily in this manner are \( \theta = \pi/2 \) (positive \( y \)-axis), \( \theta = \pi \) (negative \( x \)-axis), and \( \theta = 3\pi/2 \) (negative \( y \)-axis).

Checklist of Key Ideas:

- polar coordinate system
- angular coordinate, radial coordinate
- radian measure
- converting between rectangular and polar coordinates
- symmetry tests in polar coordinates
- circles
- roses
- cardioids
- lemniscates
- limaçons
- spirals
10.3 Tangent Lines, Arc Length, And Area For Polar Curves

PURPOSE: To find the area bounded by functions that are defined either parametrically or as polar curves.

If \( r = f(\theta) \) then
\[
\begin{align*}
\frac{dy}{d\theta} &= f'(\theta) \sin \theta + f(\theta) \cos \theta \\
\frac{dx}{d\theta} &= f'(\theta) \cos \theta - f(\theta) \sin \theta
\end{align*}
\]
A similar argument can be followed for polar curves where \( r = f(\theta) \). For example, since \( x = r \cos \theta = f(\theta) \cos \theta \) and \( y = r \sin \theta = f(\theta) \sin \theta \) then the slope at a point can be found by
\[
\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}
\]
Both of these derivatives can be found using the product rule.

tangent at origin

A special situation occurs for **tangent lines at the origin**. If \( r = f(\theta_0) = 0 \) and \( f'(\theta_0) \neq 0 \) then the line \( \theta = \theta_0 \) is tangent to the curve at the origin. It is not uncommon for a polar curve to have more than one tangent line at the origin.

**CAUTION:** At points where a polar curve intersects itself, it may have more than one tangent line.

arc length

Calculating **arc length for a polar curve** is done using the arc length formula for parameterized curves (see §10.1). For example, if a curve given by \( r = f(\theta) \) is parameterized by \( x = r \cos \theta \) and \( y = r \sin \theta \) then \( x'(t) \) and \( y'(t) \) can be found using the product rule. After squaring the terms and adding them together the resulting arc length formula can be written in terms \( r = f(\theta) \):
\[
L = \int_{\theta=a}^{\theta=b} \sqrt{(f(\theta))^2 + (f'(\theta))^2} \, d\theta
\]
A similar expression may be obtained if \( f(\theta) \) is replaced by \( r \) and \( f'(\theta) \) with \( dr/d\theta \).

**CAUTION:** To find arc length, the curve should not repeat itself on the interval \( a < \theta < b \) as given by the limits of integration and the derivative \( f'(\theta) \) needs to be continuous over the same interval.

A couple of trouble spots for this formula come from the repetitive nature of polar curves and the derivative \( f'(\theta) \). If the polar curve repeats itself for any values of \( \theta \) in the interval determined by the limits of integration, then this formula will not give an accurate value. Likewise, if \( f(\theta) \) or \( f'(\theta) \) are undefined or discontinuous then the arc length obtained may not be correct. In these situations, sometimes the techniques of improper integrals (see §7.8) may be used to find the correct arc length.

area between curves

Finding **area between curves** or in bounded regions has already been discussed in rectangular coordinates. Now, the same idea will be applied to polar coordinates. One important difference between rectangular and polar coordinates are the shapes that the area is broken into. With rectangular coordinates, an area was
broken into several vertical strips (when integrating with respect to $dx$) or horizontal strips (when integrating with respect to $dy$). In polar coordinates, the region to be measured will be broken into small infinitesimal regions by rays that come out from the origin. So the area will be broken into triangular wedges.

Each wedge is assumed to have area of a sector of a circle with central angle $d\theta$. Then to find the area formula, what is needed are the limits of integration and the “radius” of each sector. The limits of integration are determined by the angles of the starting and ending ray as determined by the boundaries of the area. Sometimes this is determined by where two curves intersect. The radius is determined by the distance to the curve (usually $r = f(\theta)$) that the ray is being drawn to.

IDEA: The area formula in polar coordinates is given by

$$A = \int_{\theta = \alpha}^{\theta = \beta} \frac{1}{2} r^2 \, d\theta$$

where $r = f(\theta)$.

Symmetry can often be use to simplify the limits of integration and then multiply the resulting integral by an appropriate integer representing the symmetry. In a rose with five petals, for example, one only needs to find the area of one petal and then multiply by five. This can often reduce the complexity of the integration that is taking place.

IDEA: Symmetry of trigonometric functions can often be used with polar coordinates to simplify the limits of integration.

In the case where the area between two curves is being found, the problem should be thought of as an inner area and outer area problem. The area that is bounded by the inner function (the one closer to the origin) is subtracted from the area bounded by the outer function (which includes the inner area).

CAUTION: When two curves are involved, the inner function is closer to the origin. Which function is closer to the origin may change, however.

It is possible that over the interval of integration the inner function and outer function will switch places. In this case, either symmetry should be used if possible to simplify the problem or multiple integrals for the different regions may need to be evaluated. When multiple integrals need to be evaluated, the places where the functions intersect should be considered as possible limits of integration. Finding intersections of polar functions may be done by setting the two equations equal and solving for $\theta$. This may not give all of the places of intersection, however, so graphing the function may be helpful as well.

CAUTION: Graphing two polar curves together can help find places where they intersect that may not be found algebraically.
Checklist of Key Ideas:

- □ lines tangent at the origin
- □ arc length of a polar curve
- □ area
- □ wedges
- □ polar coordinate area formula
- □ using symmetry to write limits of integration
- □ intersections of polar graphs

10.4 Conic Sections in Calculus

PURPOSE: To define the various curves that are conic sections and to graph these curves.

Conic sections play an important role in calculus and arise naturally in many applications. Each conic section is actually defined to be the curve that results when a plane intersects a double-napped cone (opening both upwards and downwards around a central line). There are three main types of conic sections: parabolas, ellipses, and hyperbolas. Of the three types, only the ellipse forms a closed curve. The parabola is made up of one open curve and the hyperbola is made up of two curves which open in opposite directions.

IDEA: The ellipse is the only conic section that is a closed curve. The hyperbola is the only conic section made up of two curves.

Each conic section is defined as a curve in such a way that any point on the curve has some relationship in terms of distance to special points called the foci (or focus if there is only one) of the curve.

IDEA: Each point on a conic section has some sort of relationship with a point called the focus, or two points called the foci.

Some other common traits of the conic sections are listed in the following table:
<table>
<thead>
<tr>
<th></th>
<th>ellipse</th>
<th>parabola</th>
<th>hyperbola</th>
</tr>
</thead>
<tbody>
<tr>
<td># of foci</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td># of vertices</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>center?</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>axis of symmetry?</td>
<td>no</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>directrix</td>
<td>no*</td>
<td>yes</td>
<td>no*</td>
</tr>
<tr>
<td>asymptotes?</td>
<td>no</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>focal/conjugate axis?</td>
<td>no</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>minor/major axis?</td>
<td>yes</td>
<td>no</td>
<td>no</td>
</tr>
</tbody>
</table>

*all conic sections have a directrix (see §10.6)

All conic sections also have a similar relationships between the vertices, the foci and the center of the curve. In each case, all of these points occur on the same line (although the parabola does not have a center). This line is the **axis of symmetry** for the parabola, the **major axis** of the ellipse and the **focal axis** of the hyperbola.

Below are the standard equations of the conic sections.

<table>
<thead>
<tr>
<th>Conic Section</th>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parabola</td>
<td>$y^2 = \pm 4p x$</td>
<td>opens in $\pm x$ direction $(p &gt; 0)$</td>
</tr>
<tr>
<td>Ellipse</td>
<td>$x^2/a^2 + y^2/b^2 = 1$</td>
<td>major axis in $x$ direction $(a &gt; b &gt; 0)$</td>
</tr>
<tr>
<td>Hyperbola</td>
<td>$x^2/a^2 - y^2/b^2 = 1$</td>
<td>opens in $x$ direction $(a &gt; 0, b &gt; 0)$</td>
</tr>
</tbody>
</table>

Also, each of the three conic sections has a special rectangle associated with it that can aid in determining the location of the vertices, the center and other features. The **parabola's rectangle** has dimensions of $2p$ by $4p$. The vertex is at the center of the rectangle and the curve of the parabola goes through the corners of the rectangle. One of the rectangle sides of length $4p$ borders the **directrix** line. In the center of the opposite side is the **focus** of the parabola.

The **rectangle for an ellipse** has dimensions of $2a$ by $2b$. The center of the ellipse is at the center of the rectangle and the ellipse itself is contained within the rectangle. It should be noted that in the ellipse, $a$ is always greater than $b$ and both numbers are positive. The value of $a$ represents the distance between the center of the ellipse and the vertices (which occur along the major axis). A circle is a special case of an ellipse when $a = b$ and then the radius is equal to $a$ or $b$. For example, the ellipse $\frac{x^2}{4} + \frac{y^2}{9/4} = 1$ has a value of $a = 2$ and $b = 3/2$. So the ellipse is contained within a rectangle of dimensions $4 \times 3$.

**CAUTION:** In the hyperbola, $a$ does not have to be bigger than $b$ although $a$ will always measure the distance from the center to a vertex.
The rectangle for a hyperbola has dimensions of $2a$ by $2b$, as for the ellipse, although $a$ is not always larger than $b$. In the case of the hyperbola, $a$ is also the distance between the center of the hyperbola and the vertices. The graph of the hyperbola touches its rectangle only at the vertices. Otherwise, the curves of the hyperbola occur entirely on the outside of the rectangle. If the diagonals of the rectangle are extended as lines, then these are the lines that serve as asymptotes for the branches of the hyperbola. The direction in which a hyperbola opens can be found by looking for the $x$ and $y$ intercepts of the standard equation. For example, in the case $x^2 - y^2 = 1$, the $y$-intercept would occur when $x = 0$ which leaves the equation $-y^2 = 1$. Since this equation has no real solution, the curve does not ever touch the $y$-axis.

The ellipse and hyperbola share many similarities. First, the dimensions of both rectangles are given by $a$ and $b$. Both equations involve a squared $x$ and $y$ term. Both ellipses and hyperbolas have a center, two foci and two vertices. The primary difference between ellipses and hyperbolas is that an ellipse is a closed single curve while a hyperbola is two distinct, non-closed curves.

IDEA: Conic sections may be shifted and translated like other curves. Replacing $x$ with $x - h$ translates the conic section horizontally and replacing $y$ with $y - k$ shifts the conic section vertically.

Conic sections may be translated like other curves. By replacing $x$ with $x - h$ a conic section will be translated horizontally $|h|$ units (to the right if $h > 0$ or to the left if $h < 0$). Replacing $y$ with $y - k$ will shift the conic section vertically by $|k|$ units (shifted up if $k > 0$ or down if $k < 0$).

Checklist of Key Ideas:

- conic sections, plane intersecting a cone
- degenerate conic sections
- parabola
  - → directrix, focus, axis of symmetry, vertex
- ellipse
  - → foci, center, minor/major axes, vertices
- hyperbola
  - → foci, branches, center, focal/conjugate axes, vertices
- standard equations
- translating curves
- reflection properties of conic sections
10.5 Rotation of Axes; Second-Degree Equations

**PURPOSE:** To investigate conic sections and quadratic equations that are rotated.

In the previous section, all the conic sections were aligned along either the \( x \)- or \( y \)-axes. That is, the line that contained the vertices and the foci was either parallel to the \( x \)-axis or the \( y \)-axis. In this section, this generally is not the case.

A general quadratic equation is a way of representing a conic section that may not be aligned along one of the coordinate axes. A general quadratic equation will have the following form.

\[
Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0
\]

The term \( Bxy \) is called the **cross-product term**. In the case where \( B \neq 0 \) the resulting conic section will not be aligned along the \( x \)- or \( y \)-axis.

**IDEA:** A general quadratic equation where \( B \neq 0 \) is rotated by an angle of \( \theta \) from the \( xy \)-coordinate axes where \( \cot(2\theta) = (A - C)/B \).

When \( B \neq 0 \), the conic section will be aligned along a coordinate system with \((x', y')\) **coordinates**. The new \( x' \)-axis and \( y' \)-axis are rotated from the \( x \)- and \( y \)-axes by an angle of \( \theta \). The quadratic equation may be represented in the new coordinate system by first finding \( \theta \) and then determining \( x' \) and \( y' \) in terms of \( x \), \( y \), and \( \theta \).

**IDEA:** To eliminate the cross-product term, find \( \theta \) and then rewrite \( x \) and \( y \) as \( x' \) and \( y' \).

**IDEA:** \( x' \) and \( y' \) can usually be found without calculating \( \theta \) explicitly by writing \( \cos \theta \) and \( \sin \theta \) in terms of \( \cos(2\theta) \).

If it is not important to know the actual value of \( \theta \), then \( x' \) and \( y' \) may still be calculated. Using the value of \( \cot(2\theta) \), an appropriate right triangle for \( 2\theta \) may be constructed from which a value for \( \cos(2\theta) \) may be obtained. Then \( \cos \theta \) and \( \sin \theta \) may be calculated using \( \cos(2\theta) \).

**Checklist of Key Ideas:**

- ☐ quadratic equations in \( x \) and \( y \)
- ☐ cross-product terms
- ☐ rotating axes
- ☐ rotation equations
- ☐ rotated conic sections
10.6 Conic Sections in Polar Coordinates

PURPOSE: To use polar coordinates to describe conic sections and to define the eccentricity of a conic section.

Conic sections may also be described in polar coordinates. The polar equations for conic sections have the form of 
\[ r = \frac{ed}{1 \pm e \cos \theta} \]  
and 
\[ r = \frac{ed}{1 \pm e \sin \theta} \]  
where \( e \) is the eccentricity of the conic section and \( d \) is the distance from the directrix of the conic section to the pole (or the origin). Any point on a curve that is a conic section is some distance from the focus and the directrix. The ratio of these two distances is called the eccentricity.

IDEA: The eccentricity of a conic section is 
\[ e = \frac{d_1}{d_2} \]  
where \( d_1 \) is the distance from a point to the focus and \( d_2 \) is the distance from the same point to the directrix. Eccentricity for conic sections is constant.

Knowing the eccentricity and the distance of the directrix from the pole, a conic section will be in one of the following forms:

<table>
<thead>
<tr>
<th>equation</th>
<th>directrix location ((d &gt; 0))</th>
</tr>
</thead>
</table>
| \( r = \frac{ed}{1 + e \cos \theta} \) \[| line at \( x = d \) \]
| \( r = \frac{ed}{1 - e \cos \theta} \) \[| line at \( x = -d \) \]
| \( r = \frac{ed}{1 + \sin \theta} \) \[| line at \( y = d \) \]
| \( r = \frac{ed}{1 - \sin \theta} \) \[| line at \( y = -d \) \]

CAUTION: The focus that is nearer to the directrix is assumed to be at the pole.

All conic sections can be represented using these equations. Which conic section is represented depends upon the value of the eccentricity. If \( 0 < e < 1 \) then it is an ellipse. If \( e = 1 \) then it is a parabola. Otherwise, if \( e > 1 \) then it is a hyperbola.

IDEA: The eccentricity of the conic sections obey the following relationships:

| ellipse | \( 0 < e < 1 \) |
| parabola | \( e = 1 \) |
| hyperbola | \( e > 1 \) |

From \( e \) and \( d \) together with the location of the directrix, the rest of the important information can be found about a conic section. For example, the line that contains the vertices and the focus will be perpendicular to the directrix. In the case of the parabola, the value \( p \) will be equal to \( d \).

For the ellipse and the hyperbola, the values \( r_0 \) and \( r_1 \) need to be determined in order to find \( a, b \) and \( c \). Recall that \( c \) is the distance from the center to the focus. In the case of the ellipse, the focus that is closer to the directrix is in between the
center and one of the vertices. The **distance from this focus to the closer vertex** is called $r_0$ and the **distance to the farther vertex** is $r_1$. The values of $r_0$ and $r_1$ are determined in the same way for a hyperbola although it should be noted that the closer vertex is in between the focus and the center.

**IDEA:** The values of $r_0$ and $r_1$ for the ellipse and the hyperbola are found by finding $\theta$ so that $(r, \theta)$ gives the location of the two vertices.

**CAUTION:** Finding values for $r_0$ and $r_1$ assumes that the focus closest to the directrix is located at the pole. Finding $r_0$ and $r_1$ means finding the value of $\theta$ so that the polar coordinate $(r, \theta)$ is one of the vertices. When both vertices are found, $r_0$ and $r_1$ can be calculated. The value of $\theta$ used will depend upon which line the center, the vertices and the foci are located on. If the conic section is oriented along the $x$-axis then the values of $\theta$ used would be $\theta = 0$ and $\theta = \pi$. Which point is the near vertex is dependent upon where the directrix is located.

Then for an ellipse, $a$, $b$ and $c$ can be found using the following formulas:

$$a = (r_1 + r_0)/2, \quad b = \sqrt{r_0 r_1}, \quad c = (r_1 - r_0)/2.$$  

For a hyperbola, $a$, $b$ and $c$ can be found using the following formulas.

$$a = (r_1 - r_0)/2, \quad b = \sqrt{r_0 r_1}, \quad c = (r_1 + r_0)/2.$$  

**Checklist of Key Ideas:**

- focus
- directrix
- eccentricity
- using eccentricity to identify conic sections
- polar equations of conic sections
- Kepler’s Laws of orbits, areas and periods
Chapter 10 Sample Tests

Section 10.1

1. Find \(dy/dx\) if \(x\) and \(y\) are parameterized by
\[
\begin{align*}
\{x &= 3t^2, \ y = 9t\} \\
(a) \ & \frac{3}{2t} \\
(b) \ & \frac{2t}{3} \\
(c) \ & 6t \\
(d) \ & \frac{3t}{2}
\end{align*}
\]
2. If a curve is parameterized by \(x = \sin t\) and \(y = 2\cos t\) then find \(dy/dx\).
\[
\begin{align*}
(a) \ & 2\tan(t) \\
(b) \ & 2\cot(t) \\
(c) \ & -2\tan(t) \\
(d) \ & -2\cos(t)
\end{align*}
\]
3. Find \(dy/dx\) if \(x = e^{2t}\) and \(y = t\).
\[
\begin{align*}
(a) \ & \frac{e^{-2t}}{2} \\
(b) \ & 2e^t \\
(c) \ & \frac{t}{2e^t} \\
(d) \ & 2e^{2t}
\end{align*}
\]
4. Answer true or false. If \(x = t^3\) and \(y = t^2 - 2\), then \(d^2y/dx^2 = -\frac{1}{3t}\)
5. Answer true or false. If \(x = \sin t\) and \(y = \cos t\) then \(d^2y/dx^2 = \cot t\).
6. Find the value of \(t\) for which the tangent line to the curve parameterized by \(\{x = t^4, \ y = 3t^2 - 2t + 6\}\) is horizontal.
\[
\begin{align*}
(a) \ & t = 1/3 \\
(b) \ & t = 0 \\
(c) \ & t = 2/3 \\
(d) \ & t = 8
\end{align*}
\]
7. Find the value(s) of \(t\) for which the tangent line(s) to the curve parameterized by \(\{x = 3\sin(t) + 8, \ y = 5t^2 + 7\}\) is/are horizontal.
\[
\begin{align*}
(a) \ & t = \pi/2, 3\pi/2 \\
(b) \ & t = 0 \\
(c) \ & t = -3/5
\end{align*}
\]
8. Find the value(s) of \(t\) for which the tangent line(s) to the curve parameterized by \(\{x = 2e^t - 5, \ y = 7t^2 + 3t + 1\}\) is/are horizontal.
\[
\begin{align*}
(a) \ & t = 1 \\
(b) \ & t = -3/14 \\
(c) \ & t = 3/2 \\
(d) \ & t = 1, -3/2
\end{align*}
\]
9. Find the value(s) of \(t\) for which the tangent line(s) to the curve parameterized by \(\{x = t^3 - t^2 - 5t, \ y = t^4 + 2\}\) is/are horizontal.
\[
\begin{align*}
(a) \ & t = 0 \\
(b) \ & t = 5/2 \\
(c) \ & t = 5 \\
(d) \ & t = 0, 5/2
\end{align*}
\]
10. If \(0 < t < 2\pi\), then find the value(s) of \(t\) for which the tangent line(s) to the curve parameterized by \(\{x = 6t^{3/2}, \ y = 4\sin t\}\) is/are horizontal.
\[
\begin{align*}
(a) \ & t = 0 \\
(b) \ & t = 0, \pi/2 \\
(c) \ & t = \pi/2, 3\pi/2 \\
(d) \ & t = 0, \pi/2, 3\pi/2
\end{align*}
\]

Section 10.2

1. Answer true or false. To plot \((6, \pi/4)\) in polar coordinates, go out 6 units from the pole to the right, then rotate \(\pi/4\) clockwise.
2. Find the rectangular coordinates of the polar point \((1, \pi/4)\).
\[
\begin{align*}
(a) \ & (\sqrt{2}, \sqrt{2}) \\
(b) \ & (\sqrt{2}/2, \sqrt{2}/2) \\
(c) \ & (2\sqrt{2}, 2\sqrt{2}) \\
(d) \ & (4\sqrt{2}/2, 4\sqrt{2})
\end{align*}
\]
3. Find the rectangular coordinates of the polar point \((5, -\pi/2)\).
\[
\begin{align*}
(a) \ & (5,0) \\
(b) \ & (0,5) \\
(c) \ & (-5,0) \\
(d) \ & (0,-5)
\end{align*}
\]
4. Find the rectangular coordinates of the polar point \((-4, -\pi/4)\).
\[
\begin{align*}
(a) \ & (-2\sqrt{2}, -2\sqrt{2})
\end{align*}
\]
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(b) \((2\sqrt{2}, 2\sqrt{2})\)
(c) \((-2\sqrt{2}, 2\sqrt{2})\)
(d) \((2\sqrt{2}, -2\sqrt{2})\)

5. Use a calculating utility to approximate the polar coordinates of the point \((7,3)\).
(a) \((58, 1.1903)\)
(b) \((7.6158, 0.4049)\)
(c) \((58, 1.1659)\)
(d) \((7.6158, 1.1659)\)

6. Use a calculating utility to approximate the polar coordinates of the point \((-2, -5)\).
(a) \((29, 4.3319)\)
(b) \((5.3852, 4.3319)\)
(c) \((58, 1.19031)\)
(d) \((5.3319, 1.1903)\)

7. Describe the curve \(\theta = \pi/2\).
(a) a vertical line
(b) a horizontal line
(c) a circle
(d) a semicircle

8. Describe the curve \(r = 4 \cos (\theta)\).
(a) a circle left of the origin
(b) a circle above the origin
(c) a circle right of the origin
(d) a circle below the origin

9. Describe the curve \(r = 8 \sin (\theta)\).
(a) a circle left of the origin
(b) a circle above the origin
(c) a circle right of the origin
(d) a circle below the origin

10. What is the radius of the circle \(r = 8 \sin (\theta)\)?
(a) 8
(b) 16
(c) 4
(d) 1

11. How many petals does the rose \(r = 7 \sin (3\theta)\) have?
(a) 1
(b) 4
(c) 3
(d) 6

12. Describe the curve \(r = 10 + 10 \sin (\theta)\).
(a) limacon with inner loop
(b) cardioid
(c) dimpled limacon
(d) convex limacon

13. Describe the curve \(r = 4 + 6 \cos (\theta)\).
(a) limacon with inner loop
(b) cardioid
(c) dimpled limacon
(d) convex limacon

14. Describe the curve \(r = 2 + 8 \cos (\theta)\).
(a) limacon with inner loop
(b) cardioid
(c) dimpled limacon
(d) convex limacon

15. Answer true or false. The graph of \(r = 7\theta\) in polar coordinates gives an Archimedean spiral.

Section 10.3

1. Find the approximate area of the region enclosed by \(r = 4 - 4 \cos (\theta)\).
(a) 75.40
(b) 56.55
(c) 18.85
(d) 9.42

2. Find the approximate area of the region enclosed by \(r = 4 - 4 \sin (\theta)\).
(a) 75.40
(b) 56.55
(c) 18.85
(d) 9.42

3. Find the approximate area of the region enclosed by \(r = 2 - 6 \cos (\theta)\) from \(\theta = 0\) to \(\theta = \pi/2\).
(a) 5.28
(b) 58.56
(c) 183.96
(d) 23.00

4. Find the approximate area of the region enclosed by \(r = 2 - 6 \sin (\theta)\) from \(\theta = 0\) to \(\theta = \pi/2\).
5. Answer true or false. The approximate area of the region bounded by the curve \( r = 3 \cos(2\theta) \) from \( \theta = 3\pi/2 \) to \( \theta = 2\pi \) is 3.53.

6. Answer true or false. The area between the circle \( r = 5 \) and the curve \( r = 2 + 2 \cos(\theta) \) is \( \pi/2 \).

7. Answer true or false. The area of one petal of \( r = \sin(2\theta) \) is given by \( \int_{\pi/2}^{\pi/2} 0.5 \sin(2\theta) \, d\theta \).

8. Answer true or false. The area in one petal of \( r = 2 \sin(6\theta) \) is given by \( \int_{\pi/3}^{\pi/3} 2(\sin(6\theta))^2 \, d\theta \).

9. Answer true or false. The area in all of the petals of \( r = \cos(6\theta) \) is given by \( \int_{\pi/3}^{\pi/3} 3(\sin(6\theta))^2 \, d\theta \).

10. Find the approximate area of the region bounded by \( r = 6\theta \) from \( \theta = 0 \) to \( \theta = \pi \).
    
    (a) 3.1
    (b) 2.6
    (c) 186.04
    (d) 10.3

11. Find the approximate area of the region bounded by \( r = 10\theta \) from \( \theta = 0 \) to \( \theta = 2\pi \).
    
    (a) 41.034
    (b) 2.067
    (c) 8.268
    (d) 4.134

12. Answer true or false. The region bounded between \( r = 2 \cos(\theta) \) and \( r = 2 \sin(\theta) \) is given by
    
    \[
    \int_{\pi/4}^{\pi/4} 2(\sin(\theta) - \cos(\theta))^2 \, d\theta.
    \]

13. Find the approximate area bounded by \( r = 6 - 4 \cos(\theta) \) from \( \theta = \pi \) to \( \theta = 3\pi/2 \).
    
    (a) 12.6
    (b) 29.3
    (c) 117.1
    (d) 58.6

14. Find the approximate area of the region bounded by \( r = 6 - 4 \cos(\theta) \) from \( \theta = \pi/2 \) to \( \theta = \pi \).

15. Find the approximate area bounded by \( r = 6 - 3 \sin(\theta) \) from \( \theta = \pi \) to \( \theta = 3\pi/2 \).
    
    (a) 12.6
    (b) 29.3
    (c) 117.1
    (d) 58.6

16. Answer true or false. If \( r = 7 \sin(\theta) \), then the tangent line to the curve at the origin is \( \theta = 0 \).

17. Find the arc length of the spiral \( r = e^{5\theta} \) between \( \theta = 0 \) and \( \theta = 1 \).
    
    (a) \( \frac{\sqrt{26}}{5} (e - 1) \)
    (b) \( \frac{2}{5} (e - 1) \)
    (c) \( \frac{\sqrt{26}}{5} (e^5 - 1) \)
    (d) \( \frac{2}{5} (e^5 - 1) \)

18. Find the arc length of the spiral \( r = \sin(\theta) \) between \( \theta = 0 \) and \( \theta = 2\pi \).
    
    (a) 2
    (b) \( 2\pi \)
    (c) \( 2\sqrt{2} \pi \)
    (d) \( 2\sqrt{2} \)

19. Answer true or false. The arc length of the curve \( r = \sin(2\theta) \) between \( \theta = 0 \) and \( \theta = \pi \) is \( \pi \).

20. Answer true or false. The arc length of the curve \( r = 8\theta \) between \( \theta = 0 \) and \( \theta = \pi \) is 8\pi.

Section 10.4

1. The vertex of the parabola \( y^2 = 9x \) is located at
    
    (a) (1,9)
    (b) (0,0)
    (c) (9,1)
    (d) (1,1)

2. The vertex of the parabola \((y - 6)^2 = 3(x - 2)\) is located at
    
    (a) (−2, −6)
    (b) (2,6)
    (c) (−3, −6)
3. A parabola has a vertex at (4, 5) and a directrix of $x = 0$. The focus of the parabola is located at
(a) $(4, 0)$
(b) $(4, 10)$
(c) $(8, 5)$
(d) $(8, 10)$

4. The graph of the parabola $x = 9y^2 + 2$ opens along
(a) the positive $x$-axis.
(b) the negative $x$-axis.
(c) the positive $y$-axis.
(d) the negative $y$-axis.

5. What are the ends of the minor axis for the ellipse
$$\frac{x^2}{36} + \frac{y^2}{4} = 1?$$
(a) $(6, 0)$ and $(-6, 0)$
(b) $(36, 0)$ and $(-36, 0)$
(c) $(0, 6)$ and $(0, -6)$
(d) $(0, 2)$ and $(0, -2)$

6. Answer true or false. The foci of $\frac{x^2}{49} + \frac{y^2}{81} = 1$ are $(7, 0)$ and $(-7, 0)$.

7. The foci of the ellipse $\frac{x^2}{81} + \frac{y^2}{49} = 1$ are located at
(a) $(-4\sqrt{2}, 0)$ and $(4\sqrt{2}, 0)$
(b) $(-2\sqrt{65}, 0)$ and $(2\sqrt{65}, 0)$
(c) $(-9, 0)$ and $(9, 0)$
(d) $(0, -7)$ and $(0, 7)$

8. Answer true or false. The foci of the ellipse $\frac{x^2}{64} + \frac{y^2}{36} = 1$ are $(8, 0)$ and $(-8, 0)$.

9. Answer true or false. The foci of the hyperbola $\frac{x^2}{81} - \frac{y^2}{16} = 1$ are $(0, -\sqrt{97})$ and $(0, \sqrt{97})$.

10. Answer true or false. The foci of the hyperbola $\frac{x^2}{64} - \frac{y^2}{36} = 1$ are $(10, 0)$ and $(-10, 0)$.

11. Answer true or false. The hyperbola $7y^2 - 12x^2 = 84$ opens along the $y$-axis in both the positive and negative directions.
12. Answer true or false. $7x^2 + y^2 = 7$ has a vertical major axis.
13. Answer true or false. The vertex of $y = 2x^2 + 8$ is located at $(-2, 0)$.
14. Answer true or false. The vertex of $y = x^2 - 9$ is located at $(0, -9)$.
15. Answer true or false. The vertex of $x = 5y^2 + 12$ is located at $(5/12, 0)$.

Section 10.5

1. What type of conic section is the graph of $xy = 12$?
   (a) parabola
   (b) circle
   (c) ellipse
   (d) hyperbola

2. The graph $x^2 + 6xy - 3y^2 = 0$ is what type of conic section?
   (a) parabola
   (b) circle
   (c) hyperbola
   (d) ellipse

3. The graph $x^2 + 5xy - 7y^2 + 25 = 0$ is what type of conic section?
   (a) parabola
   (b) circle
   (c) ellipse
   (d) hyperbola

4. The graph of $3xy = 5$ is a(n)
   (a) parabola
   (b) circle
   (c) ellipse
   (d) hyperbola

5. The graph $4x^2 - 4y^2 = 0$ is best described as which of the following?
   (a) hyperbola
   (b) circle
   (c) line
   (d) pair of intersecting lines

6. Answer true or false. $x^2 + 7xy + 2y^2 + 7x + 5y - 2 = 0$ represents an ellipse.

7. Answer true or false. $x^2 - 5xy + 6y^2 + 5x - 3y + 4 = 0$ represents a pair of intersecting lines.

8. Answer true or false. $6x^2 - 6y^2 + 8 = 0$ represents a pair of intersecting lines.

9. Answer true or false. $x^2 + 3xy + 2y^2 = 0$ represents a single point.

10. Answer true or false. $x^2 + 3xy + 2y^2 - 6 = 0$ represents a circle.
Section 10.6

1. The eccentricity of \( r = \frac{8}{1 + 2 \sin(\theta)} \) is
   (a) 4
   (b) 1
   (c) 2
   (d) 6

2. The eccentricity of \( r = \frac{8}{4 + 8 \cos(\theta)} \) is
   (a) 8
   (b) 4
   (c) 2
   (d) 1

3. The eccentricity of \( r = \frac{3}{5 + 15 \sin(\theta)} \) is
   (a) 12
   (b) 4
   (c) 2
   (d) 3

4. Answer true or false. \( r = \frac{5}{1 - 3 \cos(\theta)} \) has its directrix left of the pole.

5. Write the equation of the ellipse that has \( e = 3 \) and directrix \( x = 1 \).
   (a) \( r = \frac{3}{1 + 3 \cos(\theta)} \)
   (b) \( r = \frac{3}{1 - 3 \cos(\theta)} \)
   (c) \( r = \frac{3}{1 + 3 \sin(\theta)} \)
   (d) \( r = \frac{3}{1 - 3 \sin(\theta)} \)

6. The graph of \( r = \frac{6}{5 - 2 \sin(\theta)} \) is
   (a) parabola
   (b) ellipse
   (c) circle
   (d) hyperbola

7. The graph of \( r = \frac{10}{2 + 4 \cos(\theta)} \) is
   (a) parabola
   (b) ellipse
   (c) circle
   (d) hyperbola

8. The graph of \( r = \frac{7}{6 + 4 \cos(\theta)} \) is
   (a) parabola
   (b) ellipse
   (c) circle
   (d) hyperbola

9. Answer true or false. The graph of \( r = \frac{1}{5 - \cos(\theta)} \) orients horizontally.

10. Answer true or false. The graph of \( r = \frac{3}{1 - 6 \sin(\theta)} \) is a hyperbola that opens left and right.

11. Answer true or false. The graph of \( r = \frac{6}{1 - \sin(\theta)} \) is a parabola that opens to the left.

12. Answer true or false. The graph of \( r = \frac{4}{1 + \cos(\theta)} \) is a parabola that opens to the left.

13. Answer true or false. The graph of \( r = \frac{1}{6 - 3 \sin(\theta)} \) is a parabola that opens upward.

14. Answer true or false. The graph of \( r = \frac{10}{6 - 6 \sin(\theta)} \) is a hyperbola oriented along the vertical axis in both the positive and negative directions.

15. A small planet is found 8 times as far from the sun as the earth. What is its period?
   (a) 22.6 years
   (b) 64 years
   (c) 32 years
   (d) 4 years

Chapter 10 Test

1. Find the rectangular coordinates of the polar point \((-4, \pi/4)\).
   (a) \((-\sqrt{2}, \sqrt{2})\)
   (b) \((-\sqrt{2}/2, \sqrt{2}/2)\)
   (c) \((-2\sqrt{2}, -2\sqrt{2})\)
   (d) \((-4\sqrt{2}, -4\sqrt{2})\)

2. Use a calculating utility to approximate the polar coordinates of the rectangular point \((6, 8)\).
   (a) \((10, 0.9273)\)
   (b) \((100, 0.9273)\)
   (c) \((10, 0.6435)\)
3. Which of the following best describes the curve \( r = 8 \cos(\theta) \)?
   (a) a circle left of the origin
   (b) a circle above the origin
   (c) a circle right of the origin
   (d) a circle below the origin

4. What is the radius of the circle \( r = 24 \cos(\theta) \)?
   (a) 24
   (b) 48
   (c) 12
   (d) 1

5. How many petals does the rose \( r = 6 \sin(3\theta) \) have?
   (a) 1
   (b) 4
   (c) 3
   (d) 6

6. Which of the following statements best describes the curve \( r = 10 + 5 \cos(\theta) \)?
   (a) a limacon with an inner loop
   (b) a cardioid
   (c) a dimpled limacon
   (d) a convex limacon

7. Answer true or false. \( r = 7/\theta \) is the graph of a hyperbolic spiral.

8. Find \( dy/dx \) if a curve is parameterized by
   \[ \{x = -2 \cos(t), y = -2 \sin(t)\} \]
   (a) \( \tan(t) \)
   (b) \( \cot(t) \)
   (c) \( -\tan(t) \)
   (d) \( -\cot(t) \)

9. Answer true or false. If \( x = \sqrt{\sin(t)} \) and \( y = \sqrt{\cos(t)} \) then \( \frac{d^2y}{dx^2} = \cos(t) \).

10. Suppose that a curve is parameterized by
    \[ \{x = \sin(t), y = 3t^2 + 10\} \]
    Find the value(s) of \( t \) for which the tangent line(s) to the curve is/are horizontal.
    (a) \( t = \pi/2, 3\pi/2 \)
## Chapter 10: Answers to Sample Tests

### Section 10.1
1. a 2. c 3. a 4. false 5. false 6. a 7. b 8. b
9. a 10. c

### Section 10.2
1. false 2. b 3. d 4. c 5. b 6. b 7. a 8. c
9. b 10. c 11. c 12. b 13. a 14. a 15. true

### Section 10.3
1. a 2. a 3. a 4. c 5. true 6. false 7. false 8. false
17. c 18. b 19. false 20. false

### Section 10.4
1. b 2. b 3. c 4. a 5. d 6. false 7. a 8. false

### Section 10.5
9. false 10. false

### Section 10.6
1. c 2. c 3. d 4. true 5. a 6. b 7. d 8. b

### Chapter 10 Test
1. c 2. a 3. c 4. c 5. c 6. d 7. true 8. d
17. d 18. a 19. c 20. true