Solutions to Exercises, Section 1.1

For Exercises 1–8, assume that $f$ and $g$ are functions completely defined by the following tables:

\[
\begin{array}{c|c} 
 x & f(x) \\
 \hline 
 3 & 13 \\
 4 & -5 \\
 6 & \frac{3}{5} \\
 7.3 & -5
\end{array}
\quad
\begin{array}{c|c} 
 x & g(x) \\
 \hline 
 3 & 3 \\
 8 & \sqrt{7} \\
 8.4 & \sqrt{7} \\
 12.1 & -\frac{2}{7}
\end{array}
\]

1. Evaluate $f(6)$.

**SOLUTION** Looking at the table, we see that $f(6) = \frac{3}{5}$. 
3. What is the domain of $f$?

**SOLUTION** The domain of $f$ is the set of numbers in the first column of the table defining $f$. Thus the domain of $f$ is the set \{3, 4, 6, 7.3\}.
5. What is the range of $f$?

SOLUTION The range of $f$ is the set of numbers that appear in the second column of the table defining $f$. Numbers that appear more than once in the second column need to be listed only once when finding the range. Thus the range of $f$ is the set $\{13, -5, \frac{3}{2}\}$. 
7. Find two different values of $x$ such that $f(x) = -5$.

**SOLUTION** Looking at the table, we see that $f(4) = -5$ and $f(7.3) = -5$. 
9. Find all functions (displayed as tables) whose domain is the set \{2, 9\} and whose range is the set \{4, 6\}.

**SOLUTION** Because we seek functions \(f\) whose domain is the set \{2, 9\}, the first column of the table for any such function must have 2 appear once and must have 9 appear once. In other words, the table must start like this

\[
\begin{array}{c|c}
  x & f(x) \\
  \hline
  2 & \text{or this} \\
  9 & 2 \\
\end{array}
\]

The order of the rows in a table that define a function do not matter. For convenience, we choose the first possibility above.

Because the range must be the set \{4, 6\}, the second column must contain 4 and 6. There are only two slots in which to put these numbers in the first table above, and thus each one must appear exactly once in the second column. Thus there are only two functions whose domain is the set \{2, 9\} and whose range is the set \{4, 6\}; these functions are given by the following two tables:

\[
\begin{array}{c|c}
  x & f(x) \\
  \hline
  2 & 4 \\
  9 & 6 \\
\end{array}
\]

The first function above is the function \(f\) defined by \(f(2) = 4\) and \(f(9) = 6\); the second function above is the function \(f\) defined by \(f(2) = 6\) and \(f(9) = 4\).
11. Find all functions (displayed as tables) whose domain is \{1, 2, 4\} and whose range is \{-2, 1, \sqrt{3}\}.

**SOLUTION**  Because we seek functions \(f\) whose domain is \{1, 2, 4\}, the first column of the table for any such function must have 1 appear once, must have 2 appear once, and must have 4 appear once. The order of the rows in a table that define a function do not matter. For convenience, we put the first column in numerical order 1, 2, 4.

Because the range must be \{-2, 1, \sqrt{3}\}, the second column must contain \(-2, 1, \sqrt{3}\). There are only three slots in which to put these three numbers, and thus each one must appear exactly once in the second column. There are six ways in which these three numbers can be ordered. Thus the six functions whose domain is \{1, 2, 4\} and whose range is \{-2, 1, \sqrt{3}\} are given by the following tables:

\[
\begin{array}{c|c} x & f(x) \\ \hline 1 & -2 \\ 2 & 1 \\ 4 & \sqrt{3} \\
\end{array} \quad \begin{array}{c|c} x & f(x) \\ \hline 1 & -2 \\ 2 & \sqrt{3} \\ 4 & 1 \\
\end{array} \quad \begin{array}{c|c} x & f(x) \\ \hline 1 & 1 \\ 2 & \sqrt{3} \\ 4 & -2 \\
\end{array}
\]
13. Find all functions (displayed as tables) whose domain is \{3, 5, 9\} and whose range is \{2, 4\}.

**SOLUTION** Because we seek functions \(f\) whose domain is \{3, 5, 9\}, the first column of the table for any such function must have 3 appear once, must have 5 appear once, and must have 9 appear once. The order of the rows in a table that define a function do not matter. For convenience, we put the first column in numerical order 3, 5, 9.

Because the range must be \{2, 4\}, the second column must contain 2 and 4. There are three slots in which to put these three numbers, and thus one of them must be repeated once. There are six ways to do this. Thus the six functions whose domain is \{3, 5, 9\} and whose range is \{2, 4\} are given by the following tables:

\[
\begin{array}{c|c} x & f(x) \\ \hline 3 & 2 \\ 5 & 2 \\ 9 & 4 \\
\end{array} \quad \begin{array}{c|c} x & f(x) \\ \hline 3 & 2 \\ 5 & 4 \\ 9 & 2 \\
\end{array} \quad \begin{array}{c|c} x & f(x) \\ \hline 3 & 4 \\ 5 & 2 \\ 9 & 4 \\
\end{array}
\]

\[
\begin{array}{c|c} x & f(x) \\ \hline 3 & 4 \\ 5 & 4 \\ 9 & 2 \\
\end{array} \quad \begin{array}{c|c} x & f(x) \\ \hline 3 & 4 \\ 5 & 2 \\ 9 & 4 \\
\end{array} \quad \begin{array}{c|c} x & f(x) \\ \hline 3 & 2 \\ 5 & 4 \\ 9 & 4 \\
\end{array}
\]
For Exercises 15–26, assume that

\[ f(x) = \frac{x + 2}{x^2 + 1} \]

for every real number \( x \). Evaluate and simplify each of the following expressions.

15. \( f(0) \)

**SOLUTION**  \( f(0) = \frac{0+2}{0^2+1} = \frac{2}{1} = 2 \)
17. \( f(-1) \)

**SOLUTION**  
\[
f(-1) = \frac{-1 + 2}{(-1)^2 + 1} = \frac{1}{1 + 1} = \frac{1}{2}
\]
19. $f(2a)$

**SOLUTION**

\[
f(2a) = \frac{2a + 2}{(2a)^2 + 1} = \frac{2a + 2}{4a^2 + 1}
\]
21. $f(2a + 1)$

**SOLUTION**

$$f(2a + 1) = \frac{(2a + 1) + 2}{(2a + 1)^2 + 1} = \frac{2a + 3}{4a^2 + 4a + 2}$$
23. $f(x^2 + 1)$

**SOLUTION**

$$f(x^2 + 1) = \frac{(x^2 + 1) + 2}{(x^2 + 1)^2 + 1} = \frac{x^2 + 3}{x^4 + 2x^2 + 2}$$
25. \( f\left(\frac{a}{b} - 1\right) \)

**SOLUTION** We have

\[
f\left(\frac{a}{b} - 1\right) = \frac{\left(\frac{a}{b} - 1\right) + 2}{\left(\frac{a}{b} - 1\right)^2 + 1} = \frac{\frac{a}{b} + 1}{\frac{a^2}{b^2} - 2\frac{a}{b} + 2}
\]

\[
= \frac{ab + b^2}{a^2 - 2ab + 2b^2}
\]

where the last expression was obtained by multiplying the numerator and denominator of the previous expression by \(b^2\).
For Exercises 27–32, assume that

\[ g(x) = \frac{x - 1}{x + 2}. \]

27. Find a number \( b \) such that \( g(b) = 4 \).

**SOLUTION** We want to find a number \( b \) such that

\[ \frac{b - 1}{b + 2} = 4. \]

Multiply both sides of the equation above by \( b + 2 \), getting

\[ b - 1 = 4b + 8. \]

Now solve this equation for \( b \), getting \( b = -3 \).
29. Evaluate and simplify the expression \( \frac{g(x) - g(2)}{x - 2} \).

**SOLUTION** We begin by evaluating the numerator:

\[
g(x) - g(2) = \frac{x - 1}{x + 2} - \frac{1}{4}
\]

\[
= \frac{4(x - 1) - (x + 2)}{4(x + 2)}
\]

\[
= \frac{4x - 4 - x - 2}{4(x + 2)}
\]

\[
= \frac{3x - 6}{4(x + 2)}
\]

\[
= \frac{3(x - 2)}{4(x + 2)}.
\]

Thus

\[
\frac{g(x) - g(2)}{x - 2} = \frac{3(x - 2)}{4(x + 2)} \cdot \frac{1}{x - 2}
\]

\[
= \frac{3}{4(x + 2)}.
\]
31. **Evaluate and simplify the expression** \( \frac{g(a+t)-g(a)}{t} \).

**SOLUTION** We begin by computing the numerator:

\[
g(a + t) - g(a) = (a + t) - 1 \quad \frac{a - 1}{a + 2}
\]

\[
= \frac{(a + t - 1)(a + 2) - (a - 1)(a + t + 2)}{(a + t + 2)(a + 2)}
\]

\[
= \frac{3t}{(a + t + 2)(a + 2)}.
\]

Thus

\[
\frac{g(a + t) - g(a)}{t} = \frac{3}{(a + t + 2)(a + 2)}.
\]
For Exercises 33–40, assume that \( f \) is the function defined by

\[
f(x) = \begin{cases} 
2x + 9 & \text{if } x < 0 \\
3x - 10 & \text{if } x \geq 0.
\end{cases}
\]

33. Evaluate \( f(1) \).

**SOLUTION** Because \( 1 \geq 0 \), we have

\[
f(1) = 3 \cdot 1 - 10 = -7.
\]
35. Evaluate $f(-3)$.

**SOLUTION** Because $-3 < 0$, we have

$$f(-3) = 2(-3) + 9 = 3.$$
37. Evaluate $f(|x|+1)$.

**SOLUTION** Because $|x| + 1 \geq 1 > 0$, we have

$$f(|x|+1) = 3(|x| + 1) - 10 = 3|x| - 7.$$
39. Find two different values of $x$ such that $f(x) = 0$.

**Solution**  If $x < 0$, then $f(x) = 2x + 9$. We want to find $x$ such that $f(x) = 0$, which means that we need to solve the equation $2x + 9 = 0$ and hope that the solution satisfies $x < 0$. Subtracting 9 from both sides of $2x + 9 = 0$ and then dividing both sides by 2 gives $x = -\frac{9}{2}$. This value of $x$ satisfies the inequality $x < 0$, and we do indeed have $f(-\frac{9}{2}) = 0$.

If $x \geq 0$, then $f(x) = 3x - 10$. We want to find $x$ such that $f(x) = 0$, which means that we need to solve the equation $3x - 10 = 0$ and hope that the solution satisfies $x \geq 0$. Adding 10 to both sides of $3x - 10 = 0$ and then dividing both sides by 3 gives $x = \frac{10}{3}$. This value of $x$ satisfies the inequality $x \geq 0$, and we do indeed have $f(\frac{10}{3}) = 0$. 
For Exercises 41–44, find a number $b$ such that the function $f$ equals the function $g$.

41. The function $f$ has domain the set of positive numbers and is defined by $f(x) = 5x^2 - 7$; the function $g$ has domain $(b, \infty)$ and is defined by $g(x) = 5x^2 - 7$.

**SOLUTION** For two functions to be equal, they must at least have the same domain. Because the domain of $f$ is the set of positive numbers, which equals the interval $(0, \infty)$, we must have $b = 0$. 
43. Both \( f \) and \( g \) have domain \( \{3, 5\} \), with \( f \) defined on this domain by the formula \( f(x) = x^2 - 3 \) and \( g \) defined on this domain by the formula \( g(x) = \frac{18}{x} + b(x - 3) \).

**SOLUTION** Note that

\[
f(3) = 3^2 - 3 = 6 \quad \text{and} \quad f(5) = 5^2 - 3 = 22.
\]

Also,

\[
g(3) = \frac{18}{3} + b(3 - 3) = 6 \quad \text{and} \quad g(5) = \frac{18}{5} + 2b.
\]

Thus regardless of the choice of \( b \), we have \( f(3) = g(3) \). To make the function \( f \) equal the function \( g \), we must also have \( f(5) = g(5) \), which means that we must have

\[
22 = \frac{18}{5} + 2b.
\]

Solving this equation for \( b \), we get \( b = \frac{46}{5} \).
For Exercises 45–50, a formula has been given defining a function $f$ but no domain has been specified. Find the domain of each function $f$, assuming that the domain is the set of real numbers for which the formula makes sense and produces a real number.

45. $f(x) = \frac{2x+1}{3x-4}$

**SOLUTION** The formula above does not make sense when $3x - 4 = 0$, which would lead to division by 0. The equation $3x - 4 = 0$ is equivalent to $x = \frac{4}{3}$. Thus the domain of $f$ is the set of real numbers not equal to $\frac{4}{3}$. In other words, the domain of $f$ equals $\{x : x \neq \frac{4}{3}\}$, which could also be written as $(-\infty, \frac{4}{3}) \cup (\frac{4}{3}, \infty)$. 
47. \( f(x) = \frac{\sqrt{x-5}}{x-7} \)

**SOLUTION** The formula above does not make sense when \( x < 5 \) because we cannot take the square root of a negative number. The formula above also does not make sense when \( x = 7 \), which would lead to division by 0. Thus the domain of \( f \) is the set of real numbers greater than or equal to 5 and not equal to 7. In other words, the domain of \( f \) equals \( \{ x : x \geq 5 \text{ and } x \neq 7 \} \), which could also be written as \( [5, 7) \cup (7, \infty) \).
49. \( f(x) = \sqrt{|x - 6|} - 1 \)

**SOLUTION** Because we cannot take the square root of a negative number, we must have \(|x - 6| - 1 \geq 0\). This inequality is equivalent to \(|x - 6| \geq 1\), which means that \(x - 6 \geq 1\) or \(x - 6 \leq -1\). Adding 6 to both sides of these inequalities, we see that the formula above makes sense only when \(x \geq 7\) or \(x \leq 5\). In other words, the domain of \(f\) equals \(\{x : x \leq 5 \text{ or } x \geq 7\}\), which could also be written as \((-\infty, 5] \cup [7, \infty)\).
For Exercises 51–56, suppose $h$ is defined by

$$h(t) = |t| + 1.$$ 

51. What is the range of $h$ if the domain of $h$ is the interval $(1, 4]$?

**SOLUTION** For each number $t$ in the interval $(1, 4]$, we have $h(t) = t + 1$. Thus the range of $h$ is obtained by adding 1 to each number in the interval $(1, 4]$. This implies that the range of $h$ is the interval $(2, 5]$. 


53. What is the range of \( h \) if the domain of \( h \) is the interval \([-3, 5]\)?

**SOLUTION** For each number \( t \) in the interval \([-3, 0]\), we have \( h(t) = -t + 1 \), and for each number \( t \) in the interval \([0, 5]\) we have \( h(t) = t + 1 \). Thus the range of \( h \) consists of the numbers obtained by multiplying each number in the interval \([-3, 0]\) by \(-1\) and then adding 1 (this produces the interval \((1, 4]\)), along with the numbers obtained by adding 1 to each number in the interval \([0, 5]\) (this produces the interval \([1, 6]\)). This implies that the range of \( h \) is the interval \([1, 6]\).
55. What is the range of \( h \) if the domain of \( h \) is the set of positive numbers?

**SOLUTION**  For each positive number \( t \) we have \( h(t) = t + 1 \). Thus the range of \( h \) is the set obtained by adding 1 to each positive number. Hence the range of \( h \) is the interval \((1, \infty)\).
For Exercises 1–8, give the coordinates of the specified point using the figure below:

1. **A**

**SOLUTION**  To get to the point $A$ starting at the origin, we must move 3 units right and 2 units up. Thus $A$ has coordinates $(3,2)$.

Numbers obtained from a figure should be considered approximations. Thus the actual coordinates of $A$ might be $(3.01,1.98)$. 
3. C

**SOLUTION** To get to the point C starting at the origin, we must move 1 unit left and 2 units up. Thus C has coordinates (−1, 2).
5. \( E \)

**SOLUTION**  To get to the point \( E \) starting at the origin, we must move 3 units left and 2 units down. Thus \( E \) has coordinates \((-3, -2)\).
7. $G$

**SOLUTION**  To get to the point $G$ starting at the origin, we must move 1 unit right and 2 units down. Thus $G$ has coordinates $(1, -2)$. 
9. Sketch a coordinate plane showing the following four points, their coordinates, and the rectangles determined by each point (as in Example 1): (1, 2), (−2, 2), (−3, −1), (2, −3).

SOLUTION
11. Sketch the graph of the function $f$ whose domain is the set of five numbers \{-2, -1, 0, 1, 2\} and whose values are defined by the following table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

**SOLUTION**

![Graph of the function $f$]
For Exercises 13–18, assume that $g$ and $h$ are the functions completely defined by the tables below:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g(x)$</th>
<th>$x$</th>
<th>$h(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>−3</td>
<td>−1</td>
<td>−4</td>
<td>2</td>
</tr>
<tr>
<td>−1</td>
<td>1</td>
<td>−2</td>
<td>−3</td>
</tr>
<tr>
<td>1</td>
<td>2.5</td>
<td>2</td>
<td>−1.5</td>
</tr>
<tr>
<td>3</td>
<td>−2</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

13. What is the domain of $g$?

**Solution** The domain of $g$ is the set of numbers in the first column of the table defining $g$. Thus the domain of $g$ is the set $\{-3, -1, 1, 3\}$. 
15. What is the range of $g$?

**SOLUTION** The range of $g$ is the set of numbers that appear in the second column of the table defining $g$. Thus the range of $g$ is the set \{-1, 1, 2.5, -2\}.
17. Draw the graph of \( g \).

**SOLUTION** The graph of \( g \) consists of the four points with coordinates \((-3, -1), (-1, 1), (1, 2.5), (3, -2)\), as shown below:
19. Shown below is the graph of a function $f$.

   (a) What is the domain of $f$?

   (b) What is the range of $f$?

**SOLUTION**

(a) From the figure, it appears that the domain of $f$ is [0, 4].

   The word “appears” is used here because a figure cannot provide precision. The actual domain of $f$ might be [0, 4.001] or [0, 3.99] or (0, 4).

(b) From the figure, it appears that the range of $f$ is [−3, 1].
For Exercises 21–32, assume that $f$ is the function with domain $[−4, 4]$ whose graph is shown below:

21. Estimate the value of $f(−4)$.

**SOLUTION** To estimate the value of $f(−4)$, draw a vertical line from the point $−4$ on the $x$-axis to the graph, as shown below:
Then draw a horizontal line from where the vertical line intersects the graph to the $y$-axis:

![Graph](image)

The intersection of the horizontal line with the $y$-axis gives the value of $f(-4)$. Thus we see that $f(-4) \approx -3$ (the symbol $\approx$ means "is approximately equal to", which is the best that can be done when using a graph).
23. Estimate the value of $f(-2)$.

**SOLUTION** To estimate the value of $f(-2)$, draw a vertical line from the point $-2$ on the $x$-axis to the graph, as shown below:

![Graph](image)

Then draw a horizontal line from where the vertical line intersects the graph to the $y$-axis:
The intersection of the horizontal line with the $y$-axis gives the value of $f(-2)$. Thus we see that $f(-2) \approx -1$. 
25. Estimate the value of $f(2)$.

**SOLUTION** To estimate the value of $f(2)$, draw a vertical line from the point 2 on the $x$-axis to the graph, as shown below:

Then draw a horizontal line from where the vertical line intersects the graph to the $y$-axis:
The intersection of the horizontal line with the $y$-axis gives the value of $f(2)$. Thus we see that $f(2) \approx 3$. 
27. Estimate the value of \( f(4) \).

**SOLUTION** To estimate the value of \( f(4) \), draw a vertical line from the point 4 on the \( x \)-axis to the graph, as shown below:

Then draw a horizontal line from where the vertical line intersects the graph to the \( y \)-axis:
The intersection of the horizontal line with the $y$-axis gives the value of $f(4)$. Thus we see that $f(4) \approx 5$. 
29. Estimate a number $b$ such that $f(b) = 4$.

**SOLUTION** Draw the horizontal line $y = 4$, as shown below:

Then draw a vertical line from where this horizontal line intersects the graph to the $x$-axis:

The intersection of the vertical line with the $x$-axis gives the value of $b$ such that $f(b) = 4$. Thus we see that $b \approx 3.6$. 

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Instructor's Solutions Manual, Section 1.2  
Exercise 29
31. How many values of $x$ satisfy the equation $f(x) = \frac{1}{2}$?

**SOLUTION**  Draw the horizontal line $y = \frac{1}{2}$, as shown below. This horizontal line intersects the graph in three points. Thus there exist three values of $x$ such that $f(x) = \frac{1}{2}$.
For Exercises 33–44, assume that \( g \) is the function with domain \([-4, 4]\) whose graph is shown below:

33. Estimate the value of \( g(-4) \).

**SOLUTION** To estimate the value of \( g(-4) \), draw a vertical line from the point \(-4\) on the \( x\)-axis to the graph, as shown below:
Then draw a horizontal line from where the vertical line intersects the graph to the $y$-axis:

The intersection of the horizontal line with the $y$-axis gives the value of $g(-4)$. Thus we see that $g(-4) \approx 4$. 

![Graph showing the intersection](image-url)
35. Estimate the value of $g(-2)$.

**SOLUTION** To estimate the value of $g(-2)$, draw a vertical line from the point $-2$ on the $x$-axis to the graph, as shown below:

Then draw a horizontal line from where the vertical line intersects the graph to the $y$-axis:

The intersection of the horizontal line with the $y$-axis gives the value of $g(-2)$. Thus we see that $g(-2) \approx 2$. 

37. Estimate the value of $g(2)$.

**SOLUTION** To estimate the value of $g(2)$, draw a vertical line from the point 2 on the $x$-axis to the graph, as shown below:

Then draw a horizontal line from where the vertical line intersects the graph to the $y$-axis:

The intersection of the horizontal line with the $y$-axis gives the value of $g(2)$. Thus we see that $g(2) \approx -2$. 

39. Estimate the value of $g(2.5)$.

**SOLUTION** To estimate the value of $g(2.5)$, draw a vertical line from the point 2.5 on the $x$-axis to the graph, as shown below:

![Graph](image)

Then draw a horizontal line from where the vertical line intersects the graph to the $y$-axis:

![Graph](image)

The intersection of the horizontal line with the $y$-axis gives the value of $g(2.5)$. Thus we see that $g(2.5) \approx -1.5$. 
41. Estimate a number \( b \) such that \( g(b) = 3.5 \).

**SOLUTION** Draw the horizontal line \( y = 3.5 \), as shown below:

Then draw a vertical line from where this horizontal line intersects the graph to the \( x \)-axis:

The intersection of the vertical line with the \( x \)-axis gives the value of \( b \) such that \( g(b) = 3.5 \). Thus we see that \( b \approx -3.1 \).
43. How many values of $x$ satisfy the equation $g(x) = -2$?

**SOLUTION** Draw the horizontal line $y = -2$, as shown here. This horizontal line intersects the graph in three points. Thus there exist three values of $x$ such that $g(x) = -2$. 

![Graph showing intersections with line y = -2](image)
Solutions to Exercises, Section 1.3

For Exercises 1–14, assume that $f$ is the function defined on the interval $[1, 2]$ by the formula $f(x) = \frac{4}{x^2}$. Thus the domain of $f$ is the interval $[1, 2]$, the range of $f$ is the interval $[1, 4]$, and the graph of $f$ is shown here.

The graph of $f$.

For each function $g$ described below:

(a) Sketch the graph of $g$.

(b) Find the domain of $g$ (the endpoints of this interval should be shown on the horizontal axis of your sketch of the graph of $g$).

(c) Give a formula for $g$.

(d) Find the range of $g$ (the endpoints of this interval should be shown on the vertical axis of your sketch of the graph of $g$).

1. The graph of $g$ is obtained by shifting the graph of $f$ up 1 unit.

SOLUTION
Instructor's Solutions Manual, Section 1.3 Exercise 1

(a) The graph of \( f(x) = \frac{4}{x^2} + 1 \) is shifted up 1 unit to obtain the graph of \( g(x) = \frac{4}{x^2} + 1 \).

(b) The domain of \( g \) is the same as the domain of \( f \). Thus the domain of \( g \) is the interval \([1, 2]\).

(c) Because the graph of \( g \) is obtained by shifting the graph of \( f \) up 1 unit, we have \( g(x) = f(x) + 1 \). Thus

\[
g(x) = \frac{4}{x^2} + 1
\]

for each number \( x \) in the interval \([1, 2]\).

(d) The range of \( g \) is obtained by adding 1 to each number in the range of \( f \). Thus the range of \( g \) is the interval \([2, 5]\).
3. The graph of $g$ is obtained by shifting the graph of $f$ down 3 units.

SOLUTION

![Graph showing shifting](image)

(a) The domain of $g$ is the same as the domain of $f$. Thus the domain of $g$ is the interval $[1, 2]$.

(b) Because the graph of $g$ is obtained by shifting the graph of $f$ down 3 units, we have $g(x) = f(x) - 3$. Thus

$$g(x) = \frac{4}{x^2} - 3$$

for each number $x$ in the interval $[1, 2]$.

(d) The range of $g$ is obtained by subtracting 3 from each number in the range of $f$. Thus the range of $g$ is the interval $[-2, 1]$. 
5. The graph of $g$ is obtained by shifting the graph of $f$ left 3 units.

**SOLUTION**

(b) The domain of $g$ is obtained by subtracting 3 from every number in domain of $f$. Thus the domain of $g$ is the interval $[-2, -1]$.  

(c) Because the graph of $g$ is obtained by shifting the graph of $f$ left 3 units, we have $g(x) = f(x + 3)$. Thus  

$$g(x) = \frac{4}{(x + 3)^2}$$

for each number $x$ in the interval $[-2, -1]$.  

(d) The range of $g$ is the same as the range of $f$. Thus the range of $g$ is the interval $[1, 4]$. 
7. The graph of \( g \) is obtained by shifting the graph of \( f \) right 1 unit.

**SOLUTION**

Shifting the graph of \( f \) right 1 unit gives this graph.

(a)

(b) The domain of \( g \) is obtained by adding 1 to every number in domain of \( f \). Thus the domain of \( g \) is the interval \([2, 3]\).

(c) Because the graph of \( g \) is obtained by shifting the graph of \( f \) right 1 unit, we have \( g(x) = f(x - 1) \). Thus

\[
g(x) = \frac{4}{(x - 1)^2}
\]

for each number \( x \) in the interval \([2, 3]\).

(d) The range of \( g \) is the same as the range of \( f \). Thus the range of \( g \) is the interval \([1, 4]\).
9. The graph of $g$ is obtained by vertically stretching the graph of $f$ by a factor of 2.

**SOLUTION**

Vertically stretching the graph of $f$ by a factor of 2 gives this graph.

(b) The domain of $g$ is the same as the domain of $f$. Thus the domain of $g$ is the interval $[1, 2]$.

(c) Because the graph of $g$ is obtained by vertically stretching the graph of $f$ by a factor of 2, we have $g(x) = 2f(x)$. Thus

$$g(x) = \frac{8}{x^2}$$

for each number $x$ in the interval $[1, 2]$.

(d) The range of $g$ is obtained by multiplying every number in the range of $f$ by 2. Thus the range of $g$ is the interval $[2, 8]$. 
11. The graph of $g$ is obtained by horizontally stretching the graph of $f$ by a factor of 2.

**SOLUTION**

(a) Horizontally stretching the graph of $f$ by a factor of 2 gives this graph.

(b) The domain of $g$ is obtained by multiplying every number in the domain of $f$ by 2. Thus the domain of $g$ is the interval $[2, 4]$.

(c) Because the graph of $g$ is obtained by horizontally stretching the graph of $f$ by a factor of 2, we have $g(x) = f(x/2)$. Thus

$$g(x) = \frac{4}{(x/2)^2} = \frac{16}{x^2}$$

for each number $x$ in the interval $[2, 4]$.

(d) The range of $g$ is the same as the range of $f$. Thus the range of $g$ is the interval $[1, 4]$. 
13. The graph of \( g \) is obtained by reflecting the graph of \( f \) through the horizontal axis.

**SOLUTION**

Reflecting the graph of \( f \) in the horizontal axis gives this graph.

(a) \[ g(x) = -\frac{4}{x^2} \]

for each number \( x \) in the interval \([1, 2] \).

(b) The domain of \( g \) is the same as the domain of \( f \). Thus the domain of \( g \) is the interval \([1, 2] \).

(c) Because the graph of \( g \) is obtained by reflecting the graph of \( f \) through the horizontal axis, we have \( g(x) = -f(x) \). Thus

\[
g(x) = -\frac{4}{x^2}
\]

for each number \( x \) in the interval \([1, 2] \).

(d) The range of \( g \) is obtained by multiplying every number in the range of \( f \) by \(-1 \). Thus the range of \( g \) is the interval \([-4, -1] \).
For Exercises 15–46, assume that \( f \) is a function whose domain is the interval \([1, 5]\), whose range is the interval \([1, 3]\), and whose graph is the figure below.

![The graph of \( f \).](image)

For each given function \( g \):

(a) Find the domain of \( g \).
(b) Find the range of \( g \).
(c) Sketch the graph of \( g \).

15. \( g(x) = f(x) + 1 \)

**SOLUTION**

(a) Note that \( g(x) \) is defined precisely when \( f(x) \) is defined. In other words, the function \( g \) has the same domain as \( f \). Thus the domain of \( g \) is the interval \([1, 5]\).

(b) The range of \( g \) is obtained by adding 1 to every number in the range of \( f \). Thus the range of \( g \) is the interval \([2, 4]\).
(c) The graph of \( g \), shown here, is obtained by shifting the graph of \( f \) up 1 unit.
17. \( g(x) = f(x) - 3 \)

**SOLUTION**

(a) Note that \( g(x) \) is defined precisely when \( f(x) \) is defined. In other words, the function \( g \) has the same domain as \( f \). Thus the domain of \( g \) is the interval \([1, 5]\).

(b) The range of \( g \) is obtained by subtracting 3 from each number in the range of \( f \). Thus the range of \( g \) is the interval \([-2, 0]\).

(c) The graph of \( g \), shown here, is obtained by shifting the graph of \( f \) down 3 units.
19. \( g(x) = 2f(x) \)

**SOLUTION**

(a) Note that \( g(x) \) is defined precisely when \( f(x) \) is defined. In other words, the function \( g \) has the same domain as \( f \). Thus the domain of \( g \) is the interval \([1, 5]\).

(b) The range of \( g \) is obtained by multiplying each number in the range of \( f \) by 2. Thus the range of \( g \) is the interval \([2, 6]\).

(c) The graph of \( g \), shown here, is obtained by vertically stretching the graph of \( f \) by a factor of 2.
21. \( g(x) = f(x + 2) \)

**SOLUTION**

(a) Note that \( g(x) \) is defined when \( x + 2 \) is in the interval \([1, 5]\), which means that \( x \) must be in the interval \([-1, 3]\). Thus the domain of \( g \) is the interval \([-1, 3]\).

(b) The range of \( g \) is the same as the range of \( f \). Thus the range of \( g \) is the interval \([1, 3]\).

(c) The graph of \( g \), shown here, is obtained by shifting the graph of \( f \) left 2 units.
23. \( g(x) = f(x - 1) \)

**SOLUTION**

(a) Note that \( g(x) \) is defined when \( x - 1 \) is in the interval \([1, 5]\), which means that \( x \) must be in the interval \([2, 6]\). Thus the domain of \( g \) is the interval \([2, 6]\).

(b) The range of \( g \) is the same as the range of \( f \). Thus the range of \( g \) is the interval \([1, 3]\).

(c) The graph of \( g \), shown here, is obtained by shifting the graph of \( f \) right 1 unit.
25. \( g(x) = f(2x) \)

**SOLUTION**

(a) Note that \( g(x) \) is defined when \( 2x \) is in the interval \([1, 5]\), which means that \( x \) must be in the interval \([\frac{1}{2}, \frac{5}{2}]\). Thus the domain of \( g \) is the interval \([\frac{1}{2}, \frac{5}{2}]\).

(b) The range of \( g \) is the same as the range of \( f \). Thus the range of \( g \) is the interval \([1, 3]\).

(c) The graph of \( g \), shown here, is obtained by horizontally stretching the graph of \( f \) by a factor of \( \frac{1}{2} \).
27. \( g(x) = f \left( \frac{x}{2} \right) \)

**SOLUTION**

(a) Note that \( g(x) \) is defined when \( \frac{x}{2} \) is in the interval \([1, 5]\), which means that \( x \) must be in the interval \([2, 10]\). Thus the domain of \( g \) is the interval \([2, 10]\).

(b) The range of \( g \) is the same as the range of \( f \). Thus the range of \( g \) is the interval \([1, 3]\).

(c) The graph of \( g \), shown below, is obtained by horizontally stretching the graph of \( f \) by a factor of 2.
29. \( g(x) = 3 - f(x) \)

**SOLUTION**

(a) Note that \( g(x) \) is defined precisely when \( f(x) \) is defined. In other words, the function \( g \) has the same domain as \( f \). Thus the domain of \( g \) is the interval \([1, 5]\).

(b) The range of \( g \) is obtained by multiplying each number in the range of \( f \) by \(-1\) and then adding 3. Thus the range of \( g \) is the interval \([0, 2]\).

(c) The graph of \( g \), shown here, is obtained by reflecting the graph of \( f \) through the \( x \)-axis, then shifting up by 3 units.
31. \( g(x) = -f(x - 1) \)

**SOLUTION**

(a) Note that \( g(x) \) is defined when \( x - 1 \) is in the interval \([1, 5]\), which means that \( x \) must be in the interval \([2, 6]\). Thus the domain of \( g \) is the interval \([2, 6]\).

(b) The range of \( g \) is obtained by multiplying each number in the range of \( f \) by \(-1\). Thus the range of \( g \) is the interval \([-3, -1]\).

(c) The graph of \( g \), shown here, is obtained by shifting the graph of \( f \) right 1 unit, then reflecting through the \( x \)-axis.
33. \( g(x) = f(x + 1) + 2 \)

**SOLUTION**

(a) Note that \( g(x) \) is defined when \( x + 1 \) is in the interval \( [1, 5] \), which means that \( x \) must be in the interval \( [0, 4] \). Thus the domain of \( g \) is the interval \([0, 4]\).

(b) The range of \( g \) is obtained by adding 2 to each number in the range of \( f \). Thus the range of \( g \) is the interval \([3, 5]\).

(c) The graph of \( g \), shown here, is obtained by shifting the graph of \( f \) left 1 unit, then shifting up by 2 units.
35. \( g(x) = f(2x) + 1 \)

**SOLUTION**

(a) Note that \( g(x) \) is defined when \( 2x \) is in the interval \([1, 5]\), which means that \( x \) must be in the interval \([\frac{1}{2}, \frac{5}{2}]\). Thus the domain of \( g \) is the interval \([\frac{1}{2}, \frac{5}{2}]\).

(b) The range of \( g \) is obtained by adding 1 to each number in the range of \( f \). Thus the range of \( g \) is the interval \([2, 4]\).

(c) The graph of \( g \), shown here, is obtained by horizontally stretching the graph of \( f \) by a factor of \( \frac{1}{2} \), then shifting up by 1 unit.
37. \( g(x) = f(2x + 1) \)

**SOLUTION**

(a) Note that \( g(x) \) is defined when \( 2x + 1 \) is in the interval \([1, 5]\), which means that \( x \) must be in the interval \([0, 2]\) (find one endpoint of this interval by solving the equation \( 2x + 1 = 1 \); find the other endpoint by solving the equation \( 2x + 1 = 5 \)). Thus the domain of \( g \) is the interval \([0, 2]\).

(b) The range of \( g \) equals the range of \( f \). Thus the range of \( g \) is the interval \([1, 3]\).

(c) Define a function \( h \) by \( h(x) = f(x + 1) \). The graph of \( h \) is obtained by shifting the graph of \( f \) left 1 unit. Note that \( g(x) = h(2x) \). Thus the graph of \( g \) is obtained by horizontally stretching the graph of \( h \) by a factor of \( \frac{1}{2} \).

Putting this together, we see that the graph of \( g \), shown here, is obtained by shifting the graph of \( f \) left 1 unit, then horizontally stretching by a factor of \( \frac{1}{2} \).
39. \( g(x) = 2f(\frac{x}{2} + 1) \)

**SOLUTION**

(a) Note that \( g(x) \) is defined when \( \frac{x}{2} + 1 \) is in the interval \([1, 5]\), which means that \( x \) must be in the interval \([0, 8]\) (find one endpoint of this interval by solving the equation \( \frac{x}{2} + 1 = 1 \); find the other endpoint by solving the equation \( \frac{x}{2} + 1 = 5 \)). Thus the domain of \( g \) is the interval \([0, 8]\).

(b) The range of \( g \) is obtained by multiplying each number in the range of \( f \) by 2. Thus the range of \( g \) is the interval \([2, 6]\).

(c) Define a function \( h \) by \( h(x) = f(x + 1) \). The graph of \( h \) is obtained by shifting the graph of \( f \) left 1 unit. Note that \( g(x) = 2h(\frac{x}{2}) \). Thus the graph of \( g \) is obtained from the graph of \( h \) by stretching horizontally by a factor of 2 and stretching vertically by a factor of 2.

Putting this together, we see that the graph of \( g \), shown below, is obtained by shifting the graph of \( f \) left 1 unit, then stretching horizontally by a factor of 2 and stretching vertically by a factor of 2.
41. $g(x) = 2f\left(\frac{x}{2} + 1\right) - 3$

**SOLUTION**

(a) Note that $g(x)$ is defined when $\frac{x}{2} + 1$ is in the interval $[1, 5]$, which means that $x$ must be in the interval $[0, 8]$ (find one endpoint of this interval by solving the equation $\frac{x}{2} + 1 = 1$; find the other endpoint by solving the equation $\frac{x}{2} + 1 = 5$). Thus the domain of $g$ is the interval $[0, 8]$.

(b) The range of $g$ is obtained by multiplying each number in the range of $f$ by 2 and then subtracting 3. Thus the range of $g$ is the interval $[-1, 3]$.

(c) The graph of $g$, shown below, is obtained by shifting the graph obtained in the solution to Exercise 39 down 3 units.
43. \( g(x) = 2f\left(\frac{x}{2} + 3\right) \)

**SOLUTION**

(a) Note that \( g(x) \) is defined when \( \frac{x}{2} + 3 \) is in the interval \([1, 5]\), which means that \( x \) must be in the interval \([-4, 4]\) (find one endpoint of this interval by solving the equation \( \frac{x}{2} + 3 = 1 \); find the other endpoint by solving the equation \( \frac{x}{2} + 3 = 5 \)). Thus the domain of \( g \) is the interval \([-4, 4]\).

(b) The range of \( g \) is obtained by multiplying each number in the range of \( f \) by 2. Thus the range of \( g \) is the interval \([2, 6]\).

(c) Define a function \( h \) by \( h(x) = f(x + 3) \). The graph of \( h \) is obtained by shifting the graph of \( f \) left 3 units. Note that \( g(x) = 2h\left(\frac{x}{2}\right) \). Thus the graph of \( g \) is obtained from the graph of \( h \) by stretching horizontally by a factor of 2 and stretching vertically by a factor of 2.

Putting this together, we see that the graph of \( g \), shown below, is obtained by shifting the graph of \( f \) left 3 units, then stretching horizontally by a factor of 2 and stretching vertically by a factor of 2.
45. \( g(x) = 6 - 2f(\frac{x}{2} + 3) \)

**SOLUTION**

(a) Note that \( g(x) \) is defined when \( \frac{x}{2} + 3 \) is in the interval \([1, 5]\), which means that \( x \) must be in the interval \([-4, 4]\) (find one endpoint of this interval by solving the equation \( \frac{x}{2} + 3 = 1 \); find the other endpoint by solving the equation \( \frac{x}{2} + 3 = 5 \)). Thus the domain of \( g \) is the interval \([-4, 4]\).

(b) The range of \( g \) is obtained by multiplying each number in the range of \( f \) by \(-2\) and then adding 6. Thus to find the range of \( g \), consider the equation \( z = 6 - 2y \). As \( y \) varies over the range of \( f \) (which is the interval \([1, 3]\)), \( z \) will vary over the range of \( g \). When \( y = 1 \), we see that \( z = 4 \). When \( y = 3 \), we see that \( z = 0 \). Thus the range of \( g \) is the interval \([0, 4]\).

(c) The graph of \( g \), shown below, is obtained by reflecting through the \( x \)-axis the graph obtained in the solution to Exercise 43, then shifting up by 6 units.
For Exercises 47–50, suppose \( f \) is a function whose domain is the interval \([-5, 5]\) and that

\[
f(x) = \frac{x}{x + 3}
\]

for every \( x \) in the interval \([0, 5]\).

47. Suppose \( f \) is an even function. Evaluate \( f(-2) \).

**SOLUTION** Because 2 is in the interval \([0, 5]\), we can use the formula above to evaluate \( f(2) \). We have

\[
f(2) = \frac{2}{2+3} = \frac{2}{5}.
\]

Because \( f \) is an even function, we have

\[
f(-2) = f(2) = \frac{2}{5}.
\]
49. Suppose $f$ is an odd function. Evaluate $f(-2)$.

**SOLUTION**  Because $f$ is an odd function, we have

$$f(-2) = -f(2) = -\frac{2}{5}.$$
Solutions to Exercises, Section 1.4

For Exercises 1–10, evaluate the indicated expression assuming that \( f \), \( g \), and \( h \) are the functions completely defined by the tables below:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( x )</th>
<th>( g(x) )</th>
<th>( x )</th>
<th>( h(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

1. \((f \circ g)(1)\)

**SOLUTION** \((f \circ g)(1) = f(g(1)) = f(2) = 1\)
3. \((g \circ f)(1)\)

**SOLUTION** \((g \circ f)(1) = g(f(1)) = g(4) = 3\)
5. \((f \circ f)(2)\)

**SOLUTION** \((f \circ f)(2) = f(f(2)) = f(1) = 4\)
7. \((g \circ g)(4)\)

**SOLUTION** \((g \circ g)(4) = g(g(4)) = g(3) = 1\)
9. \((f \circ g \circ h)(2)\)

**SOLUTION**

\[
(f \circ g \circ h)(2) = f(g(h(2)))
\]

\[
= f(g(3)) = f(1) = 4
\]
For Exercises 11–24, evaluate the indicated expression assuming that

\[ f(x) = \sqrt{x}, \quad g(x) = \frac{x + 1}{x + 2}, \quad h(x) = |x - 1|. \]

11. \((f \circ g)(4)\)

**SOLUTION**

\[
(f \circ g)(4) = f(g(4)) = f\left(\frac{4 + 1}{4 + 2}\right) = f\left(\frac{5}{6}\right) = \sqrt{\frac{5}{6}}
\]
13. \((g \circ f)(4)\)

**SOLUTION**

\[
(g \circ f)(4) = g(f(4))
\]

\[
= g(\sqrt{4}) = g(2) = \frac{2 + 1}{2} = \frac{3}{4}
\]
15. \((f \circ h)(-3)\)

**SOLUTION**

\[
(f \circ h)(-3) = f(h(-3)) = f(|-3 - 1|) = f(|-4|) = f(4) = \sqrt{4} = 2
\]
17. \((f \circ g \circ h)(0)\)

**SOLUTION**

\[
(f \circ g \circ h)(0) = f(g(h(0)))
\]

\[
= f(g(1)) = f\left(\frac{2}{3}\right) = \sqrt{\frac{2}{3}}
\]
19. \((f \circ g)(0.23)\)

**SOLUTION**

\[
(f \circ g)(0.23) = f(g(0.23)) = f\left(\frac{0.23 + 1}{0.23 + 2}\right)
\]

\[
\approx f(0.55157) = \sqrt{0.55157} \approx 0.74268
\]
21. \((g \circ f)(0.23)\)

**SOLUTION**

\[
(g \circ f)(0.23) = g(f(0.23)) = g(\sqrt{0.23})
\]

\[
\approx g(0.47958) = \frac{0.47958 + 1}{0.47958 + 2}
\]

\[
\approx 0.59671
\]
23. \((h \circ f)(0.3)\)

**SOLUTION**

\[(h \circ f)(0.3) = h(f(0.3)) = h(\sqrt{0.3})\]

\[\approx h(0.547723) = |0.547723 - 1|\]

\[= | - 0.452277| = 0.452277\]
In Exercises 25–30, for the given functions \( f \) and \( g \) find formulas for (a) \( f \circ g \) and (b) \( g \circ f \). Simplify your results as much as possible.

25. \( f(x) = x^2 + 1, \quad g(x) = \frac{1}{x} \)

SOLUTION

(a) \[
(f \circ g)(x) = f(g(x))
= f\left(\frac{1}{x}\right)
= \left(\frac{1}{x}\right)^2 + 1
= \frac{1}{x^2} + 1
\]

(b) \[
(g \circ f)(x) = g(f(x))
= g(x^2 + 1)
= \frac{1}{x^2 + 1}
\]
27. \( f(x) = \frac{x - 1}{x + 1}, \quad g(x) = x^2 + 2 \)

**SOLUTION**

(a) 

\[
(f \circ g)(x) = f(g(x)) = f(x^2 + 2) = \frac{(x^2 + 2) - 1}{(x^2 + 2) + 1} = \frac{x^2 + 1}{x^2 + 3}
\]

(b) 

\[
(g \circ f)(x) = g(f(x)) = g\left(\frac{x - 1}{x + 1}\right) = \left(\frac{x - 1}{x + 1}\right)^2 + 2
\]
29. \( f(x) = \frac{x - 1}{x^2 + 1}, \quad g(x) = \frac{x + 3}{x + 4} \)

SOLUTION

(a) We have

\[(f \circ g)(x) = f(g(x))\]

\[= f\left(\frac{x + 3}{x + 4}\right)\]

\[= \frac{\frac{x + 3}{x + 4} - 1}{\left(\frac{x + 3}{x + 4}\right)^2 + 1}\]

\[= \frac{(x + 3)(x + 4) - (x + 4)^2}{(x + 3)^2 + (x + 4)^2}\]

\[= \frac{x^2 + 7x + 12 - x^2 - 8x - 16}{x^2 + 6x + 9 + x^2 + 8x + 16}\]

\[= \frac{-x - 4}{2x^2 + 14x + 25}.\]

In going from the third line above to the fourth line, both numerator and denominator were multiplied by \((x + 4)^2\).

(b) We have
\((g \circ f)(x) = g(f(x))\)

\[ = g\left(\frac{x - 1}{x^2 + 1}\right) \]

\[ = \frac{x - 1}{x^2 + 1} + 3 \]

\[ = \frac{x - 1 + 3(x^2 + 1)}{x^2 + 1} + 4 \]

\[ = \frac{x - 1 + 3x^2 + 1}{x^2 + 1} + 4 \]

\[ = \frac{3x^2 + x + 2}{4x^2 + x + 3}. \]

In going from the third line above to the fourth line, both numerator and denominator were multiplied by \(x^2 + 1\).
31. Find a number $b$ such that $f \circ g = g \circ f$, where $f(x) = 2x + b$ and $g(x) = 3x + 4$.

**SOLUTION** We will compute $(f \circ g)(x)$ and $(g \circ f)(x)$, then set those two expressions equal to each other and solve for $b$. We begin with $(f \circ g)(x)$:

$$(f \circ g)(x) = f(g(x)) = f(3x + 4)$$

$$= 2(3x + 4) + b = 6x + 8 + b.$$

Next we compute $(g \circ f)(x)$:

$$(g \circ f)(x) = g(f(x)) = g(2x + b)$$

$$= 3(2x + b) + 4 = 6x + 3b + 4.$$

Looking at the expressions for $(f \circ g)(x)$ and $(g \circ f)(x)$, we see that they will equal each other if

$$8 + b = 3b + 4.$$

Solving this equation for $b$, we get $b = 2$. 
33. Suppose

\[ h(x) = \left( \frac{x^2 + 1}{x - 1} - 1 \right)^3. \]

(a) If \( f(x) = x^3 \), then find a function \( g \) such that \( h = f \circ g \).

(b) If \( f(x) = (x - 1)^3 \), then find a function \( g \) such that \( h = f \circ g \).

**SOLUTION**

(a) We want the following equation to hold: \( h(x) = f(g(x)) \). Replacing \( h \) and \( f \) with the formulas for them, we have

\[ \left( \frac{x^2 + 1}{x - 1} - 1 \right)^3 = (g(x))^3. \]

Looking at the equation above, we see that we want to choose

\[ g(x) = \frac{x^2 + 1}{x - 1} - 1. \]

(b) We want the following equation to hold: \( h(x) = f(g(x)) \). Replacing \( h \) and \( f \) with the formulas for them, we have

\[ \left( \frac{x^2 + 1}{x - 1} - 1 \right)^3 = (g(x) - 1)^3. \]

Looking at the equation above, we see that we want to choose

\[ g(x) = \frac{x^2 + 1}{x - 1}. \]
35. Suppose

\[ h(x) = 2 + \sqrt{\frac{1}{x^2 + 1}}. \]

(a) If \( g(x) = \frac{1}{x^2 + 1} \), then find a function \( f \) such that \( h = f \circ g \).

(b) If \( g(x) = x^2 \), then find a function \( f \) such that \( h = f \circ g \).

**SOLUTION**

(a) We want the following equation to hold: \( h(x) = f(g(x)) \). Replacing \( h \) and \( g \) with the formulas for them, we have

\[ 2 + \sqrt{\frac{1}{x^2 + 1}} = f\left( \frac{1}{x^2 + 1} \right). \]

Looking at the equation above, we see that we want to choose \( f(x) = 2 + \sqrt{x} \).

(b) We want the following equation to hold: \( h(x) = f(g(x)) \). Replacing \( h \) and \( g \) with the formulas for them, we have

\[ 2 + \sqrt{\frac{1}{x^2 + 1}} = f(x^2). \]

Looking at the equation above, we see that we want to choose

\[ f(x) = 2 + \sqrt{\frac{1}{x + 1}}. \]
In Exercises 37–40, find functions $f$ and $g$, each simpler than the given function $h$, such that $h = f \circ g$.

37. $h(x) = (x^2 - 1)^2$

SOLUTION The last operation performed in the computation of $h(x)$ is squaring. Thus the most natural way to write $h$ as a composition of two functions $f$ and $g$ is to choose $f(x) = x^2$, which then suggests that we choose $g(x) = x^2 - 1$. 
39. \( h(x) = \frac{3}{2 + x^2} \)

**SOLUTION** The last operation performed in the computation of \( h(x) \) is dividing 3 by a certain expression. Thus the most natural way to write \( h \) as a composition of two functions \( f \) and \( g \) is to choose \( f(x) = \frac{3}{x} \), which then requires that we choose \( g(x) = 2 + x^2 \).
In Exercises 41–42, find functions \( f \), \( g \), and \( h \), each simpler than the given function \( T \), such that \( T = f \circ g \circ h \).

41. \( T(x) = \frac{4}{5 + x^2} \)

**SOLUTION** A good solution is to take

\[
f(x) = \frac{4}{x}, \quad g(x) = 5 + x, \quad h(x) = x^2.
\]
Solutions to Exercises, Section 1.5

For Exercises 1–8, check your answer by evaluating the appropriate function at your answer.

1. Suppose \( f(x) = 4x + 6 \). Evaluate \( f^{-1}(5) \).

   **SOLUTION**  We need to find a number \( x \) such that \( f(x) = 5 \). In other words, we need to solve the equation
   \[
   4x + 6 = 5.
   \]
   This equation has solution \( x = -\frac{1}{4} \). Thus \( f^{-1}(5) = -\frac{1}{4} \).

   **CHECK**  To check that \( f^{-1}(5) = -\frac{1}{4} \), we need to verify that \( f(-\frac{1}{4}) = 5 \).
   We have
   \[
   f(-\frac{1}{4}) = 4(-\frac{1}{4}) + 6 = 5,
   \]
   as desired.
3. Suppose \( g(x) = \frac{x + 2}{x + 1} \). Evaluate \( g^{-1}(3) \).

**SOLUTION** We need to find a number \( x \) such that \( g(x) = 3 \). In other words, we need to solve the equation

\[
\frac{x + 2}{x + 1} = 3.
\]

Multiplying both sides of this equation by \( x + 1 \) gives the equation

\[
x + 2 = 3x + 3,
\]

which has solution \( x = -\frac{1}{2} \). Thus \( g^{-1}(3) = -\frac{1}{2} \).

**CHECK** To check that \( g^{-1}(3) = -\frac{1}{2} \), we need to verify that \( g(-\frac{1}{2}) = 3 \). We have

\[
g(-\frac{1}{2}) = \frac{-\frac{1}{2} + 2}{-\frac{1}{2} + 1} = \frac{\frac{3}{2}}{\frac{1}{2}} = 3,
\]

as desired.
5. Suppose \( f(x) = 3x + 2 \). Find a formula for \( f^{-1} \).

**SOLUTION** For each number \( y \), we need to find a number \( x \) such that \( f(x) = y \). In other words, we need to solve the equation

\[
3x + 2 = y
\]

for \( x \) in terms of \( y \). Subtracting 2 from both sides of the equation above and then dividing both sides by 3 gives

\[
x = \frac{y - 2}{3}.
\]

Thus

\[
f^{-1}(y) = \frac{y - 2}{3}
\]

for every number \( y \).

**CHECK** To check that \( f^{-1}(y) = \frac{y - 2}{3} \), we need to verify that \( f\left(\frac{y - 2}{3}\right) = y \).

We have

\[
f\left(\frac{y - 2}{3}\right) = 3\left(\frac{y - 2}{3}\right) + 2 = y,
\]

as desired.
7. Suppose \( h(t) = \frac{1 + t}{2 - t} \). Find a formula for \( h^{-1} \).

**SOLUTION** For each number \( y \), we need to find a number \( t \) such that \( h(t) = y \). In other words, we need to solve the equation

\[
\frac{1 + t}{2 - t} = y
\]

for \( t \) in terms of \( y \). Multiplying both sides of this equation by \( 2 - t \) and the collecting all the terms with \( t \) on one side gives

\[
t + yt = 2y - 1.
\]

Rewriting the left side as \((1 + y)t\) and then dividing both sides by \(1 + y\) gives

\[
t = \frac{2y - 1}{1 + y}.
\]

Thus

\[
h^{-1}(y) = \frac{2y - 1}{1 + y}
\]

for every number \( y \neq -1 \).

**CHECK** To check that \( h^{-1}(y) = \frac{2y - 1}{1 + y} \), we need to verify that \( h\left(\frac{2y - 1}{1 + y}\right) = y \). We have

\[
h\left(\frac{2y - 1}{1 + y}\right) = \frac{1 + \frac{2y - 1}{1 + y}}{2 - \frac{2y - 1}{1 + y}}.
\]
Multiplying numerator and denominator of the expression on the right by $1 + \gamma$ gives

$$h\left(\frac{2\gamma - 1}{1 + \gamma}\right) = \frac{1 + \gamma + 2\gamma - 1}{2 + 2\gamma - 2\gamma + 1} = \frac{3\gamma}{3} = \gamma,$$

as desired.
9. Suppose \( f(x) = 2 + \frac{x - 5}{x + 6} \).

(a) Evaluate \( f^{-1}(4) \).
(b) Evaluate \( f(4)^{-1} \).

**SOLUTION**

(a) We need to find a number \( x \) such that \( f(x) = 4 \). In other words, we need to solve the equation

\[
2 + \frac{x - 5}{x + 6} = 4.
\]

Subtracting 2 from both sides and then multiplying both sides by \( x + 6 \) gives the equation

\[
x - 5 = 2x + 12,
\]

which has solution \( x = -17 \). Thus \( f^{-1}(4) = -17 \).

(b) Note that

\[
f(4) = 2 + \frac{4 - 5}{4 + 6} = \frac{20}{10} - \frac{1}{10} = \frac{19}{10}.
\]

Thus \( f(4)^{-1} = \frac{10}{19} \).
11. Suppose $h(x) = 5x^2 + 7$, where the domain of $h$ is the set of positive numbers. Find a formula for $h^{-1}$.

**SOLUTION** For each number $y$, we need to find a number $x$ such that $h(x) = y$. In other words, we need to solve the equation

$$5x^2 + 7 = y$$

for $x$ in terms of $y$. Subtracting 7 from both sides of the equation above and then dividing both sides by 5 and taking square roots gives

$$x = \sqrt{\frac{y - 7}{5}},$$

where we chose the positive square root because $x$ is required to be a positive number. Thus

$$h^{-1}(y) = \sqrt{\frac{y - 7}{5}}$$

for every number $y > 7$ (the restriction that $y > 7$ is required to insure that we get a positive number when evaluating the formula above).
For each of the functions \( f \) given in Exercises 13–22:

(a) Find the domain of \( f \).
(b) Find the range of \( f \).
(c) Find a formula for \( f^{-1} \).
(d) Find the domain of \( f^{-1} \).
(e) Find the range of \( f^{-1} \).

You can check your solutions to part (c) by verifying that \( f^{-1} \circ f = I \) and \( f \circ f^{-1} = I \) (recall that \( I \) is the function defined by \( I(x) = x \)).

13. \( f(x) = 3x + 5 \)

**SOLUTION**

(a) The expression \( 3x + 5 \) makes sense for all real numbers \( x \). Thus the domain of \( f \) is the set of real numbers.

(b) To find the range of \( f \), we need to find the numbers \( y \) such that

\[
    y = 3x + 5
\]

for some \( x \) is the domain of \( f \). In other words, we need to find the values of \( y \) such that the equation above can be solved for a real number \( x \). Solving the equation above for \( x \), we get

\[
    x = \frac{y - 5}{3}.
\]
The expression above on the right makes sense for every real number \( y \). Thus the range of \( f \) is the set of real numbers.

(c) The expression above shows that \( f^{-1} \) is given by the formula

\[
f^{-1}(y) = \frac{y - 5}{3}.
\]

(d) The domain of \( f^{-1} \) equals the range of \( f \). Thus the domain of \( f^{-1} \) is the set of real numbers.

(e) The range of \( f^{-1} \) equals the domain of \( f \). Thus the range of \( f^{-1} \) is the set of real numbers.
15. \( f(x) = \frac{1}{3x + 2} \)

**SOLUTION**

(a) The expression \( \frac{1}{3x + 2} \) makes sense except when \( 3x + 2 = 0 \). Solving this equation for \( x \) gives \( x = -\frac{2}{3} \). Thus the domain of \( f \) is the set \( \{ x : x \neq -\frac{2}{3} \} \).

(b) To find the range of \( f \), we need to find the numbers \( y \) such that

\[
y = \frac{1}{3x + 2}
\]

for some \( x \) is the domain of \( f \). In other words, we need to find the values of \( y \) such that the equation above can be solved for a real number \( x \neq -\frac{2}{3} \). To solve this equation for \( x \), multiply both sides by \( 3x + 2 \), getting

\[
3xy + 2y = 1.
\]

Now subtract \( 2y \) from both sides, then divide by \( 3y \), getting

\[
x = \frac{1 - 2y}{3y}.
\]

The expression above on the right makes sense for every real number \( y \neq 0 \) and produces a number \( x \neq -\frac{2}{3} \) (because the equation \(-\frac{2}{3} = \frac{1 - 2y}{3y}\) leads to nonsense, as you can verify if you try to solve it for \( y \)). Thus the range of \( f \) is the set \( \{ y : y \neq 0 \} \).
(c) The expression above shows that $f^{-1}$ is given by the formula

$$f^{-1}(y) = \frac{1 - 2y}{3y}.$$

(d) The domain of $f^{-1}$ equals the range of $f$. Thus the domain of $f^{-1}$ is the set $\{y : y \neq 0\}$.

(e) The range of $f^{-1}$ equals the domain of $f$. Thus the range of $f^{-1}$ is the set $\{x : x \neq -\frac{2}{3}\}$. 
17. \( f(x) = \frac{2x}{x + 3} \)

**SOLUTION**

(a) The expression \( \frac{2x}{x + 3} \) makes sense except when \( x = -3 \). Thus the domain of \( f \) is the set \( \{x : x \neq -3\} \).

(b) To find the range of \( f \), we need to find the numbers \( y \) such that

\[
y = \frac{2x}{x + 3}
\]

for some \( x \) is the domain of \( f \). In other words, we need to find the values of \( y \) such that the equation above can be solved for a real number \( x \neq -3 \). To solve this equation for \( x \), multiply both sides by \( x + 3 \), getting

\[
xy + 3y = 2x.
\]

Now subtract \( xy \) from both sides, getting

\[
3y = 2x - xy = x(2 - y).
\]

Dividing by \( 2 - y \) gives

\[
x = \frac{3y}{2 - y}.
\]

The expression above on the right makes sense for every real number \( y \neq 2 \) and produces a number \( x \neq -3 \) (because the equation \( -3 = \frac{3y}{2 - y} \) leads to nonsense, as you can verify if you try to solve it for \( y \)). Thus the range of \( f \) is the set \( \{y : y \neq 2\} \).
(c) The expression above shows that \( f^{-1} \) is given by the formula

\[
f^{-1}(y) = \frac{3y}{2-y}.
\]

(d) The domain of \( f^{-1} \) equals the range of \( f \). Thus the domain of \( f^{-1} \) is the set \( \{y : y \neq 2\} \).

(e) The range of \( f^{-1} \) equals the domain of \( f \). Thus the range of \( f^{-1} \) is the set \( \{x : x \neq -3\} \).
19. \( f(x) = \begin{cases} 3x & \text{if } x < 0 \\ 4x & \text{if } x \geq 0 \end{cases} \)

**SOLUTION**

(a) The expression defining \( f(x) \) makes sense for all real numbers \( x \). Thus the domain of \( f \) is the set of real numbers.

(b) To find the range of \( f \), we need to find the numbers \( y \) such that \( y = f(x) \) for some real number \( x \). From the definition of \( f \), we see that if \( y < 0 \), then \( y = f(x) \) for \( x = \frac{y}{3} \), and if \( y \geq 0 \), then \( y = f(x) \) for \( x = \frac{y}{4} \). Thus every real number \( y \) is in the range of \( f \). In other words, the range of \( f \) is the set of real numbers.

(c) From the paragraph above, we see that \( f^{-1} \) is given by the formula

\[
 f^{-1}(y) = \begin{cases} \frac{y}{3} & \text{if } y < 0 \\ \frac{y}{4} & \text{if } y \geq 0 \end{cases}.
\]

(d) The domain of \( f^{-1} \) equals the range of \( f \). Thus the domain of \( f^{-1} \) is the set of real numbers.

(e) The range of \( f^{-1} \) equals the domain of \( f \). Thus the range of \( f^{-1} \) is the set of real numbers.
21. \( f(x) = x^2 + 8 \), where the domain of \( f \) equals \((0, \infty)\).

**SOLUTION**

(a) As part of the definition of the function \( f \), the domain has been specified to be the interval \((0, \infty)\), which is the set of positive numbers.

(b) To find the range of \( f \), we need to find the numbers \( y \) such that

\[
y = x^2 + 8
\]

for some \( x \) is the domain of \( f \). In other words, we need to find the values of \( y \) such that the equation above can be solved for a positive number \( x \). To solve this equation for \( x \), subtract 8 from both sides and then take square roots of both sides, getting

\[
x = \sqrt{y - 8},
\]

where we chose the positive square root of \( y - 8 \) because \( x \) is required to be a positive number.

The expression above on the right makes sense and produces a positive number \( x \) for every number \( y > 8 \). Thus the range of \( f \) is the interval \((8, \infty)\).

(c) The expression above shows that \( f^{-1} \) is given by the formula

\[
f^{-1}(y) = \sqrt{y - 8}.
\]

(d) The domain of \( f^{-1} \) equals the range of \( f \). Thus the domain of \( f^{-1} \) is the interval \((8, \infty)\).
(e) The range of $f^{-1}$ equals the domain of $f$. Thus the range of $f^{-1}$ is the interval $(0, \infty)$, which is the set of positive numbers.
23. Suppose \( f(x) = x^5 + 2x^3 \). Which of the numbers listed below equals \( f^{-1}(8.10693) \)?

\[ 1.1, \quad 1.2, \quad 1.3, \quad 1.4 \]

[For this particular function, it is not possible to find a formula for \( f^{-1}(y) \).]

**SOLUTION** First we test whether or not \( f^{-1}(8.10693) \) equals 1.1 by checking whether or not \( f(1.1) \) equals 8.10693. Using a calculator, we find that

\[ f(1.1) = 4.27251, \]

which means that \( f^{-1}(8.10693) \neq 1.1 \).

Next we test whether or not \( f^{-1}(8.10693) \) equals 1.2 by checking whether or not \( f(1.2) \) equals 8.10693. Using a calculator, we find that

\[ f(1.2) = 5.94432, \]

which means that \( f^{-1}(8.10693) \neq 1.2 \).

Next we test whether or not \( f^{-1}(8.10693) \) equals 1.3 by checking whether or not \( f(1.3) \) equals 8.10693. Using a calculator, we find that

\[ f(1.3) = 8.10693, \]

which means that \( f^{-1}(8.10693) = 1.3 \).
In 2006 the U. S. federal income tax for a single person with taxable income $t$ dollars (this is the net income after allowable deductions) was $f(t)$ dollars, where $f$ is the function defined as follows:

$$f(t) = \begin{cases} 
0.1t & \text{if } 0 \leq t \leq 7550 \\
0.15t - 377.5 & \text{if } 7550 < t \leq 30650 \\
0.25t - 3442.5 & \text{if } 30650 < t \leq 74200 \\
0.28t - 5668.5 & \text{if } 74200 < t \leq 154800 \\
0.33t - 13408.5 & \text{if } 154800 < t \leq 336550 \\
0.35t - 20139.5 & \text{if } 336550 < t.
\end{cases}$$

Use the information above for Exercises 25–26.

25. What is the taxable income of a single person who paid $10,000 in federal taxes?

SOLUTION We need to evaluate $f^{-1}(10000)$. Letting $t = f^{-1}(10000)$, this means that we need to solve the equation $f(t) = 10000$ for $t$. Determining which formula to apply requires a bit of experimentation. Using the definition of $f$, we can calculate that $f(7550) = 755$, $f(30650) = 4220$, and $f(74200) = 15107.5$. Because 10000 is between 4220 and 15107.5, this means that $t$ is between 30650 and 74200. Thus $f(t) = 0.25t - 3442.5$. Solving the equation

$$0.25t - 3442.5 = 10000$$

for $t$, we get $t = 53770$. Thus a single person whose federal tax bill was $10,000 had a taxable income of $53,770.
27. Suppose $g(x) = x^7 + x^3$. Evaluate

$$(g^{-1}(4))^7 + (g^{-1}(4))^3 + 1.$$ 

**SOLUTION** We are asked to evaluate $g(g^{-1}(4)) + 1$. Because $g(g^{-1}(4)) = 4$, the quantity above equals 5.
Solutions to Exercises, Section 1.6

For Exercises 1–24 suppose $f$ and $g$ are functions, each of whose domain consists of four numbers, with $f$ and $g$ defined by the tables below:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$x$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

1. What is the domain of $f$?

**SOLUTION** The domain of $f$ equals the set of numbers in the left column of the table defining $f$. Thus the domain of $f$ equals $\{1, 2, 3, 4\}$. 
3. What is the range of $f$?

**SOLUTION** The range of $f$ equals the set of numbers in the right column of the table defining $f$. Thus the range of $f$ equals $\{2, 3, 4, 5\}$.
5. Sketch the graph of \( f \).

**SOLUTION** The graph of \( f \) consists of all points of the form \( (x, f(x)) \) as \( x \) varies over the domain of \( f \). Thus the graph of \( f \), shown below, consists of the four points \( (1,4) \), \( (2,5) \), \( (3,2) \), and \( (4,3) \).
7. Give the table of values for \( f^{-1} \).

**SOLUTION** The table for the inverse of a function is obtained by interchanging the two columns of the table for the function (after which one can, if desired, reorder the rows, as has been done below):

<table>
<thead>
<tr>
<th>( y )</th>
<th>( f^{-1}(y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>
9. What is the domain of $f^{-1}$?

**SOLUTION** The domain of $f^{-1}$ equals the range of $f$. Thus the domain of $f^{-1}$ is the set \{2, 3, 4, 5\}. 
11. What is the range of $f^{-1}$?

**SOLUTION** The range of $f^{-1}$ equals the domain of $f$. Thus the range of $f^{-1}$ is the set $\{1, 2, 3, 4\}$. 
13. Sketch the graph of $f^{-1}$.

**SOLUTION** The graph of $f^{-1}$ consists of all points of the form $(x, f^{-1}(x))$ as $x$ varies over the domain of $f^{-1}$. Thus the graph of $f^{-1}$, shown below, consists of the four points $(4, 1)$, $(5, 2)$, $(2, 3)$, and $(3, 4)$. 

![Graph of f^{-1}](image-url)
15. Give the table of values for $f^{-1} \circ f$.

**SOLUTION** We know that $f^{-1} \circ f$ is the identity function on the domain of $f$; thus no computations are necessary. However, because this function $f$ has only four numbers in its domain, it may be instructive to compute $(f^{-1} \circ f)(x)$ for each value of $x$ in the domain of $f$. Here is that computation:

- $(f^{-1} \circ f)(1) = f^{-1}(f(1)) = f^{-1}(4) = 1$
- $(f^{-1} \circ f)(2) = f^{-1}(f(2)) = f^{-1}(5) = 2$
- $(f^{-1} \circ f)(3) = f^{-1}(f(3)) = f^{-1}(2) = 3$
- $(f^{-1} \circ f)(4) = f^{-1}(f(4)) = f^{-1}(3) = 4$

Thus, as expected, the table of values for $f^{-1} \circ f$ is as shown below:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$(f^{-1} \circ f)(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>
17. Give the table of values for $f \circ f^{-1}$.

**SOLUTION** We know that $f \circ f^{-1}$ is the identity function on the range of $f$ (which equals the domain of $f^{-1}$); thus no computations are necessary. However, because this function $f$ has only four numbers in its range, it may be instructive to compute $(f \circ f^{-1})(y)$ for each value of $y$ in the range of $f$. Here is that computation:

\[
\begin{align*}
(f \circ f^{-1})(2) &= f(f^{-1}(2)) = f(3) = 2 \\
(f \circ f^{-1})(3) &= f(f^{-1}(3)) = f(4) = 3 \\
(f \circ f^{-1})(4) &= f(f^{-1}(4)) = f(1) = 4 \\
(f \circ f^{-1})(5) &= f(f^{-1}(5)) = f(2) = 5 \\
\end{align*}
\]

Thus, as expected, the table of values for $f \circ f^{-1}$ is as shown here.

<table>
<thead>
<tr>
<th>$y$</th>
<th>$(f \circ f^{-1})(y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>
19. Give the table of values for \( f \circ g \).

**SOLUTION** We need to compute \((f \circ g)(x)\) for every \(x\) in the domain of \(g\). Here is that computation:

\[
\begin{align*}
(f \circ g)(2) &= f(g(2)) = f(3) = 2 \\
(f \circ g)(3) &= f(g(3)) = f(2) = 5 \\
(f \circ g)(4) &= f(g(4)) = f(4) = 3 \\
(f \circ g)(5) &= f(g(5)) = f(1) = 4
\end{align*}
\]

Thus the table of values for \( f \circ g \) is as shown here.

<table>
<thead>
<tr>
<th>(x)</th>
<th>((f \circ g)(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>
21. Give the table of values for \((f \circ g)^{-1}\).

**SOLUTION** The table of values for \((f \circ g)^{-1}\) is obtained by interchanging the two columns of the table for \((f \circ g)\) (after which one can, if desired, reorder the rows, as has been done below).

<table>
<thead>
<tr>
<th>(y)</th>
<th>((f \circ g)^{-1}(y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
</tr>
</tbody>
</table>

Thus the table for \((f \circ g)^{-1}\) is as shown here.
23. Give the table of values for $g^{-1} \circ f^{-1}$.

**SOLUTION** We need to compute $(g^{-1} \circ f^{-1})(y)$ for every $y$ in the domain of $f^{-1}$. Here is that computation:

$(g^{-1} \circ f^{-1})(2) = g^{-1}(f^{-1}(2)) = g^{-1}(3) = 2$

$(g^{-1} \circ f^{-1})(3) = g^{-1}(f^{-1}(3)) = g^{-1}(4) = 4$

$(g^{-1} \circ f^{-1})(4) = g^{-1}(f^{-1}(4)) = g^{-1}(1) = 5$

$(g^{-1} \circ f^{-1})(5) = g^{-1}(f^{-1}(5)) = g^{-1}(2) = 3$

Thus the table of values for $g^{-1} \circ f^{-1}$ is as shown here.

<table>
<thead>
<tr>
<th>$y$</th>
<th>$(g^{-1} \circ f^{-1})(y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
</tr>
</tbody>
</table>
For Exercises 25–36, use the following graphs:

Here \( f \) has domain \([0, 4]\) and \( g \) has domain \((-1, 5)\).

25. What is the largest interval contained in the domain of \( f \) on which \( f \) is increasing?

**SOLUTION** As can be seen from the graph, \([3, 4]\) is the largest interval on which \( f \) is increasing.

As usual when obtaining information solely from graphs, this answer (as well as the answers to the other parts of this exercise) should be considered an approximation. An expanded graph at a finer scale might show that \([2.99, 4]\) or \([3.01, 4]\) would be a more accurate answer than \([3, 4]\).
27. Let $F$ denote the function obtained from $f$ by restricting the domain to the interval in Exercise 25. What is the domain of $F^{-1}$?

**SOLUTION** The domain of $F^{-1}$ equals the range of $F$. Because $F$ is the function $f$ with domain restricted to the interval $[3, 4]$, we see from the graph above that the range of $F$ is the interval $[-3, -2]$. Thus the domain of $F^{-1}$ is the interval $[-3, -2]$. 
29. With $F$ as in Exercise 27, what is the range of $F^{-1}$?

**SOLUTION** The range of $F^{-1}$ equals the domain of $F$. Thus the range of $F^{-1}$ is the interval $[3, 4]$. 
31. What is the largest interval contained in the domain of $f$ on which $f$ is decreasing?

**SOLUTION** As can be seen from the graph, $[0, 3]$ is the largest interval on which $f$ is decreasing.
33. Let $H$ denote the function obtained from $f$ by restricting the domain to the interval in Exercise 31. What is the domain of $H^{-1}$?

**Solution** The domain of $H^{-1}$ equals the range of $H$. Because $H$ is the function $f$ with domain restricted to the interval $[0, 3]$, we see from the graph above that the range of $H$ is the interval $[-3, 1]$. Thus the domain of $H^{-1}$ is the interval $[-3, 1]$. 
35. With $H$ as in Exercise 33, what is the range of $H^{-1}$?

**SOLUTION** The range of $H^{-1}$ equals the domain of $H$. Thus the range of $H^{-1}$ is the interval $[0, 3]$. 