Costs and Revenue
Given its demand and cost functions, a firm can calculate the break-even output level, the level of output for maximum total revenue and the level of output for maximum profit (or minimum loss).

Q. The demand and total cost functions for a monopolist are given as

Demand function: \[ P = 120 - 5Q \] (1)
Total cost function: \[ TC = 4/30Q^3 - 2Q^2 + 18Q + 60 \] (2)

(a) Derive the Total Revenue function and calculate the level of output to give maximum Total Revenue.
(b) Show the effect on price elasticity of demand as output passes through the maximum total revenue level of output.
(c) Calculate the break-even level of output.
(d) Calculate the level of output to give maximum profits (or minimum losses).

Solution
(a) Total Revenue.
Total revenue, \( TR = PQ \)
\[ = (120 - 5Q) Q \]
\[ = 120Q - 5Q^2 \] (3)
Total Revenue is maximum when the slope of the Total Revenue curve is 0, that is when \( dTR/dQ = 0 \).

\[ TR = 120Q - 5Q^2 \]
\[ dTR/dQ = 120 - 10Q \]
\[ 0 = 120 - 10Q \] at maximum
\[ 10Q = 120 \]
\[ Q = 12 \]

Total Revenue is a maximum when output is 12.
Substituting 12 into equation (3), maximum Total Revenue is 720.
To plot the Total Revenue function, first calculate a table of points: see Table 6.1C. The graph of the Total Revenue is given in Figure 6.1C.

<table>
<thead>
<tr>
<th>$Q$</th>
<th>0</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>$TR = 120Q - 5Q^2$</td>
<td>0</td>
<td>400</td>
<td>640</td>
<td>720</td>
<td>640</td>
<td>400</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 6.1C. Calculating Total Revenue curve.

Figure 6.1 C. Total Revenue Function

A monopolist is the only supplier of the good in the market. The monopolist can set the price of the good (a price setter) and supply the quantity consumers’ demand at that price.
Because a monopolist faces a downward sloping demand curve, initially, as output increases, total revenue also increases. However, beyond a certain output level, total revenue starts to decrease - the inelastic section of the demand curve has been reached and the percentage increase in demand is insufficient to offset the percentage reduction in price.

(b) Show the effect on price elasticity of demand as output passes through the maximum total revenue level of output. $Q = 12$

Solution
In chapter 3 we saw that price elasticity of demand measures the responsiveness of demand to a change in price. Price elasticity of demand, $e_d$, is calculated by $\frac{dQ}{dP} \times \frac{P}{Q}$ as follows:

$$P = 120 - 5Q$$
$$5Q = 120 - P$$
$$Q = 24 - 0.2P$$
$$\frac{dQ}{dP} = -0.2$$

$$e_d = \frac{dQ}{dP} \times \frac{P}{Q} = -0.2 \times \frac{P}{Q}$$

When $Q = 11$ (before maximum total revenue), $P = 120 - 5(11) = 65$.
$$e_d = -0.2 \times \frac{65}{11}$$
$$= -0.2 \times 5.91$$
$$= -1.18$$ (more negative than -1, elastic).

When $Q = 12$ (level of output for maximum total revenue), $P = 120 - 5(12) = 60$.
$$e_d = -0.2 \times \frac{60}{12}$$
$$= -0.2 \times 5$$
$$= -1$$ (unit elastic).

When $Q = 13$ (after maximum total revenue), $P = 120 - 5(13) = 55$.
$$e_d = -0.2 \times \frac{55}{13}$$
$$= -0.2 \times 4.23$$
$$= -0.85$$ (between -1 and 0, inelastic).

As price decreases and demand increases we move down the demand curve. Price elasticity of demand decreases, going from elastic through unit elastic, to inelastic. Below the maximum total revenue level of output, the monopolist is operating on the elastic section of the demand curve. An increase in output results in an increase in total revenue.
Beyond the maximum total revenue level of output, the monopolist is operating on the inelastic section of the demand curve. An increase in output results in a decrease in total revenue.

(c) **Calculate the Break-Even level of output.**

**Solution**

The break-even level of output occurs when

Total cost = Total revenue.

Hence

\[
\frac{4}{30}Q^3 - 2Q^2 + 18Q + 60 = 120Q - 5Q^2
\]

\[
\frac{4}{30}Q^3 + 3Q^2 - 102Q + 60 = 0
\]

The solution of such cubic equations is beyond the scope of this text. However an approximate solution may be found graphically (see Worked Example 4.11) by plotting the Total cost function and the Total cost function on the same diagram: the point(s) of intersection of the graphs gives the break-even point(s).

The table of points for each function given in Table 6.2C: the graph is plotted in Figure 6.2C.

**Table 6.2C. Points for plotting Total Revenue and Total Cost**

<table>
<thead>
<tr>
<th>Q</th>
<th>0</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>( TR = 120Q - 5Q^2 )</td>
<td>0</td>
<td>400</td>
<td>640</td>
<td>720</td>
<td>640</td>
<td>400</td>
<td>0</td>
</tr>
<tr>
<td>( TC = \frac{4}{30}Q^3 - 2Q^2 + 18Q + 60 )</td>
<td>60</td>
<td>108.53</td>
<td>144.27</td>
<td>218.40</td>
<td>382.13</td>
<td>686.67</td>
<td></td>
</tr>
</tbody>
</table>
By plotting total cost against total revenue we can see that break-even occurs at output levels of approximately 0.6 and 18.2. Below 0.6 units and above 18.2 units the firm will make a loss.

(d) Calculate the Profit Maximisation level of output.

Solution

The output decisions of a monopolist are generally designed to maximise profits. Profit maximum (or loss minimum) occurs when marginal cost equals marginal revenue. Marginal cost is the increase in total cost when output is increased by 1 unit. Marginal revenue is the increase in total revenue when output is increased by 1 unit. If the cost of making one extra unit is less than the revenue received from selling that extra unit the firm should increase output, up to the point where $MC = MR$.

Marginal cost is given by the slope of the total cost curve.
$TC = 4/30Q^3 - 2Q^2 + 18Q + 60$

$$MC = \frac{d(TC)}{dQ}$$

$$= 0.4Q^2 - 4Q + 18 \quad (4)$$

Similarly, marginal revenue is given by the slope of the total revenue curve.

$TR = 120Q - 5Q^2$

$$MR = \frac{d(TR)}{dQ}$$

$$= 120 - 10Q \quad (5)$$

When $MC = MR$

$$0.4Q^2 - 4Q + 18 = 120 - 10Q$$

$$0.4Q^2 + 6Q - 102 = 0$$

$$Q^2 + 15Q - 255 = 0$$

$$Q = \frac{-15 \pm \sqrt{(15)^2 - 4(1)(-255)}}{2}$$

$$= \frac{-15 \pm \sqrt{225 + 1020}}{2}$$

$$= \frac{-15 \pm \sqrt{1245}}{2}$$

$$= \frac{-15 \pm 35.28}{2}$$

$$= 10.14 \quad \text{(disregard the negative value of } Q)$$

Profit is at a maximum (or loss is at a minimum) when $Q = 10.14$.

The profit maximising level of output, $Q = 10.14$ is shown in Figure 6.2C
Table 6.3C  Points for plotting Demand, Marginal Cost and Marginal Revenue curves.

<table>
<thead>
<tr>
<th>Q</th>
<th>0</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
<th>24</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P = 120 - 5Q$</td>
<td>120</td>
<td>100</td>
<td>80</td>
<td>60</td>
<td>40</td>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>$MR = 120 - 10Q$</td>
<td>120</td>
<td>80</td>
<td>40</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$MC = 0.4Q^2 - 4Q + 18$</td>
<td>18</td>
<td>8.4</td>
<td>11.6</td>
<td>27.6</td>
<td>56.4</td>
<td>98</td>
<td>152.4</td>
</tr>
</tbody>
</table>

Figure 6.3C  Marginal Cost = Marginal Revenue for maximum profit.

**Profit or loss at $Q = 10.14$?**

An output level of 10.14 units, where marginal cost equals marginal revenue, will give maximum profit, or minimum loss.

To determine if an output of 10.14 units results in a profit or loss, we must compare *average cost* and *average price* at that output.

**Average cost** is equal to total cost divided by quantity.

$$AC = TC/Q$$

$$TC = 4/30Q^3 - 2Q^2 + 18Q + 60$$

$$AC = 4/30Q^2 - 2Q + 18 + 60/Q$$  … (6)
Average price is calculated from the demand function when $Q = 10.14$.

$P = 120 - 5Q = 120 - 5(10.14)$.

$P = 69.30$

As the selling price of 69.30 is greater than the average cost, 17.35, the monopolist is making supernormal profits of 51.95 per unit. Because the industry is a monopoly there are barriers to entry which prevent other firms from entering and competing away the supernormal profits.

Table 6.4C  Points for plotting Demand, Marginal Revenue, Marginal Cost and Average Cost curves.

<table>
<thead>
<tr>
<th>$Q$</th>
<th>$P = 120 - 5Q$</th>
<th>$MR = 120 - 10Q$</th>
<th>$MC = 0.4Q^2 - 4Q + 18$</th>
<th>$AC = 4/30Q^2 - 2Q + 18 + 60/Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>120</td>
<td>120</td>
<td>18</td>
<td>27.13</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>80</td>
<td>8.4</td>
<td>18.03</td>
</tr>
<tr>
<td>8</td>
<td>80</td>
<td>40</td>
<td>11.6</td>
<td>18.20</td>
</tr>
<tr>
<td>12</td>
<td>60</td>
<td>0</td>
<td>27.6</td>
<td>23.88</td>
</tr>
<tr>
<td>16</td>
<td>40</td>
<td></td>
<td>56.4</td>
<td>34.33</td>
</tr>
<tr>
<td>20</td>
<td>20</td>
<td></td>
<td>98</td>
<td>49.30</td>
</tr>
<tr>
<td>24</td>
<td>0</td>
<td></td>
<td>152.4</td>
<td></td>
</tr>
</tbody>
</table>

$AC = 17.35$ when $Q = 10.14$
Figure 6.4C. Profit Maximising Output and Supernormal Profit