Appendix M: Root Locus
Rules: Derivations

M.1 Derivation of the Behavior of the Root Locus at Infinity (Kuo, 1987)

Let the open-loop transfer function be represented as follows:

\[
KG(s)H(s) = \frac{K(s^m + a_1s^{m-1} + \ldots + a_m)}{(s^n + b_1s^{n-1} + \ldots + b_m + n)} \quad (M.1)
\]

or

\[
KG(s)H(s) = \frac{K}{(s^n + b_1s^{n-1} + \ldots + b_m + n)} \quad (M.2)
\]

Performing the indicated division in the denominator, we obtain

\[
KG(s)H(s) = \frac{K}{s^n + (b_1 - a_1)s^{n-1} + \ldots} \quad (M.3)
\]

In order for a pole of the closed-loop transfer function to exist,

\[
KG(s)H(s) = -1 \quad (M.4)
\]

Assuming large values of \( s \) that would exist as the locus moves toward infinity, Eq. (M.3) becomes

\[
s^n + (b_1 - a_1)s^{n-1} = -K \quad (M.5)
\]

Factoring out \( s^n \), Eq. (M.5) becomes

\[
s^n \left(1 + \frac{b_1 - a_1}{s}\right) = -K \quad (M.6)
\]

Taking the \( n \)th root of both sides, we have

\[
s \left(1 + \frac{b_1 - a_1}{s}\right)^{1/n} = -K^{1/n} \quad (M.7)
\]
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If the term
\[
\left(1 + \frac{b_1 - a_1}{s}\right)^{1/n}
\]
(M.8)
is expanded into an infinite series where only the first two terms are significant,\(^1\) we obtain
\[
s\left(1 + \frac{b_1 - a_1}{ns}\right) = (-K)^{1/n}
\]  
(M.9)

Distributing the factor \(S\) on the left-hand side yields
\[
s + \frac{b_1 - a_1}{n} = (-K)^{1/n}
\]  
(M.10)

Now, letting \(s = \sigma + j\omega\) and \((-K)^{1/n} = |K^{1/n}|e^{j(2k+1)\pi/n}\), where
\[
(-1)^{1/n} = e^{j(2k+1)\pi/n} = \cos\left(\frac{(2k+1)\pi}{n}\right) + j\sin\left(\frac{(2k+1)\pi}{n}\right)
\]  
(M.11)

Eq. (M.10) becomes
\[
\sigma + j\omega + \frac{b_1 - a_1}{n} = |K^{1/n}|\left[\cos\left(\frac{(2k+1)\pi}{n}\right) + j\sin\left(\frac{(2k+1)\pi}{n}\right)\right]
\]  
(M.12)

where \(k = 0, \pm 1, \pm 2, \pm 3, \ldots\) Setting the real and imaginary parts of both sides equal to each other, we obtain
\[
\sigma + \frac{b_1 - a_1}{n} = |K^{1/n}|\cos\left(\frac{(2k+1)\pi}{n}\right)
\]  
(M.13a)
\[
\omega = |K^{1/n}|\sin\left(\frac{(2k+1)\pi}{n}\right)
\]  
(M.13b)

Dividing the two equations to eliminate \(|K^{1/n}|\), we obtain
\[
\frac{\sigma + \frac{b_1 - a_1}{n}}{\omega} = \frac{\cos\left(\frac{(2k+1)\pi}{n}\right)}{\sin\left(\frac{(2k+1)\pi}{n}\right)}
\]  
(M.14)

Finally, solving for \(\omega\), we find
\[
\omega = \left[\tan\left(\frac{(2k+1)\pi}{n}\right)\right] \left[\frac{\sigma + \frac{b_1 - a_1}{n}}{\omega}\right]
\]  
(M.15)

The form of this equation is that of a straight line,
\[
\omega = M(\sigma - \sigma_0)
\]  
(M.16)

where the slope of the line, \(M\), is
\[
M = \tan\left(\frac{(2k+1)\pi}{n}\right)
\]  
(M.17)

\(^1\)This is a good approximation since \(s\) is approaching infinity for the region applicable to the derivation.
M.2 Derivation of Transition Method for Breakaway and Break-in Points

Thus, the angle of the line in radians with respect to the positive extension of the real axis is

$$\theta = \frac{(2k + 1)\pi}{n}$$

(M.18)

and the $\sigma$ intercept is

$$\sigma_0 = -\left[\frac{b_1 - a_1}{n}\right]$$

(M.19)

From the theory of equations,$^2$

$$b_1 = -\sum \text{finite poles}$$

(M.20a)

$$a_1 = -\sum \text{finite zeros}$$

(M.20b)

Also, from Eq. (M.1),

$$n = \text{number of finite poles} - \text{number of finite zeros}$$

$$= \#\text{finite poles} - \#\text{finite zeros}$$

(M.21)

By examining Eq. (M.16), we conclude that the root locus approaches a straight line as the locus approaches infinity. Further, this straight line intersects the $\sigma$ axis at

$$\sigma_0 = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}}$$

(M.22)

which is obtained by substituting Eqs. (M.20)

Let us summarize the results: The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equation of the asymptotes is given by the real-axis intercept and the angle with respect to the real axis as follows:

$$\sigma_0 = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}}$$

(M.23)

$$\theta = \frac{(2k + 1)\pi}{\#\text{finite poles} - \#\text{finite zeros}}$$

(M.24)

where $k = 0, \pm 1, \pm 2, \pm 3, \ldots$ Notice that the running index, $k$, in Eq. (M.24) yields a multiplicity of lines that account for the many branches of a root locus that approach infinity.

M.2 Derivation of Transition Method for Breakaway and Break-in Points

The transition method for finding real-breakaway and break-in points without differentiating can be derived by showing that the natural log of $1/[G(\sigma)H(\sigma)]$ has a zero derivative at the same value of $\sigma$ as $1/[G(\sigma)H(\sigma)]$ (Franklin, 1991).

$^2$Given an $n$th-order polynomial of the form $x^n + a_{n-1}x^{n-1} + \cdots$, the coefficient, $a_{n-1}$, is the negative sum of the roots of the polynomial.
Appendix M: Root Locus Rules: Derivations

We now show that if we work with the natural log we can eliminate the step of differentiation.

First find the derivative of the natural log of $1/[G(\sigma)H(\sigma)]$ and set it equal to zero. Thus,

$$\frac{d}{d\sigma} \ln \left[ \frac{1}{G(\sigma)H(\sigma)} \right] = G(\sigma)H(\sigma) \frac{d}{d\sigma} \left[ \frac{1}{G(\sigma)H(\sigma)} \right] = 0 \tag{M.25}$$

Since $G(\sigma)H(\sigma)$ is not zero at the breakaway or break-in points, letting

$$\frac{d}{d\sigma} \ln \left[ \frac{1}{G(\sigma)H(\sigma)} \right] = 0 \tag{M.26}$$

will thus yield the same value of $\sigma$ as letting

$$\frac{d}{d\sigma} \left[ \frac{1}{G(\sigma)H(\sigma)} \right] = 0 \tag{M.27}$$

Hence,

$$\frac{d}{d\sigma} \ln \left[ \frac{1}{G(\sigma)H(\sigma)} \right] = \frac{d}{d\sigma} \ln \left[ \frac{(\sigma+p_1)(\sigma+p_2)\cdots(\sigma+p_n)}{(\sigma+z_1)(\sigma+z_2)\cdots(\sigma+z_m)} \right]$$

$$= \frac{d}{d\sigma} \ln(\sigma+p_1) + \frac{d}{d\sigma} \ln(\sigma+p_2) + \cdots + \frac{d}{d\sigma} \ln(\sigma+p_n) - \frac{d}{d\sigma} \ln(\sigma+z_1) - \frac{d}{d\sigma} \ln(\sigma+z_2) - \cdots - \frac{d}{d\sigma} \ln(\sigma+z_m)$$

$$= \frac{1}{\sigma+p_1} + \frac{1}{\sigma+p_2} + \cdots + \frac{1}{\sigma+p_n} - \frac{1}{\sigma+z_1} - \frac{1}{\sigma+z_2} - \cdots - \frac{1}{\sigma+z_m} = 0 \tag{M.28}$$

or

$$\sum_{i=1}^{n} \frac{1}{\sigma+p_i} = \sum_{i=1}^{m} \frac{1}{\sigma+z_i} \tag{M.29}$$

where $z_i$ and $p_i$ are the negatives of the zero and pole values of $G(s)H(s)$, respectively. Equation (M.29) can be solved for $\sigma$, the real axis values that minimize or maximize $K$, yielding the breakaway and break-in points without differentiating.

### Bibliography

